Outline

- Lattice field theory:
  - Quantum Mechanics with Path Integral (demo)
  - Lattice ϕ^4
  - Scalar QED $\rightarrow$ QCD
- Monte Carlo
- Finite temperature: Y-M deconfinement transition
- Fermions:
  - Continuum symmetries
  - Species doubling
  - Numerical simulation
  - Finite temperature
- Finite chemical potential:
  - Expectations
  - Sign problem
  - Imaginary chemical potential

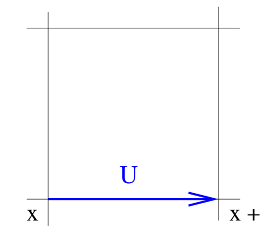
The simplest gauge theory: scalar QED

- Change $\phi(x) \in \mathcal{R}$ to $\phi(x) \in \mathcal{C}$: $\mathcal{L}_E = |\partial_\mu \phi(x)|^2 + m_0^2 |\phi(x)|^2 + \frac{g_0}{4!}|\phi(x)|^4$

- Note global $U(1)$ symmetry: $\phi(x) \to \exp(i\alpha)\phi(x)\ \forall x$, $\mathcal{L}_E$ unchanged

- **IDEA:** promote global symmetry to **local** "gauge" symmetry: $\phi(x) \to \exp(i\alpha(x))\phi(x)$

  Obstruction: $\partial_\mu \phi(x)$, or on the lattice: $\frac{1}{a}(\phi(x+\hat\mu) - \phi(x))$



- **SOLUTION:** - introduce new degrees of freedom attached to links: $U_\mu(x) \in U(1)$

  - modify derivative to covariant derivative $D_\mu \phi(x) \to \frac{1}{a}(U_\mu(x)\phi(x+\hat\mu) - \phi(x))$

  - impose transformation of links $U_\mu(x) \to \exp(i\alpha(x))U_\mu(x)\exp(-i\alpha(x+\hat\mu))$

- Only gauge-invariant observables have non-zero vev: $\phi^*(x)\ (\prod_{x \to y} U)\ \phi(y)$

And also closed [Wilson] loops: $\mathrm{Tr} \prod_{x \to x} U$

Dynamics of Wilson loops?

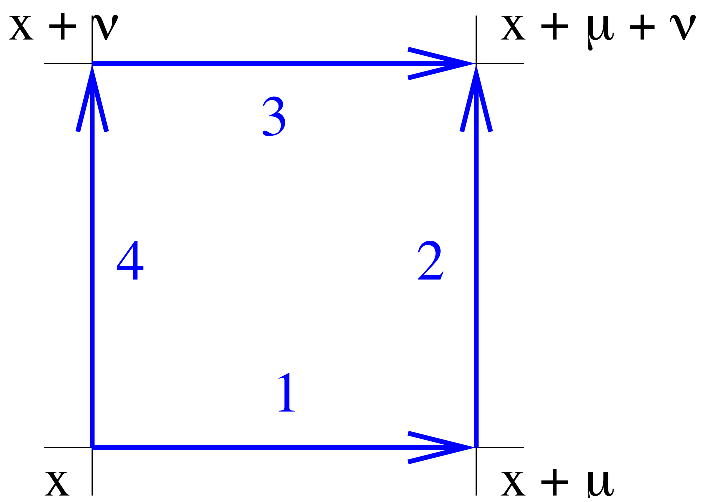
- Action S_g allowed to contain $\sum_{\text{shape}} \sum_{\text{rep. } R} c_{R,\text{shape}} \, \mathrm{Tr}_R \, W_{\text{shape}}$

  Sum over lattice translations and rotations (Lorentz symmetry)

- Simplest choice (smallest loop): "Wilson action" $S_g \propto \sum_{x,\mu\nu} \left(\mathrm{Tr}\ U_{P_{\mu\nu}}(x) + \mathrm{Tr}\ U^{-1}_{P_{\mu\nu}}(x) \right)$

$U_{P_{\mu\nu}}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U^\dagger_\mu(x+\hat{\nu}) U^\dagger_\nu(x)$ (cf. curl)

$U_{P_{\mu\nu}}(x)$ "plaquette" matrix; $U_{-\mu}(x+\hat{\mu}) = U^{-1}{}_\mu(x)$; $U^{-1} = U^\dagger$



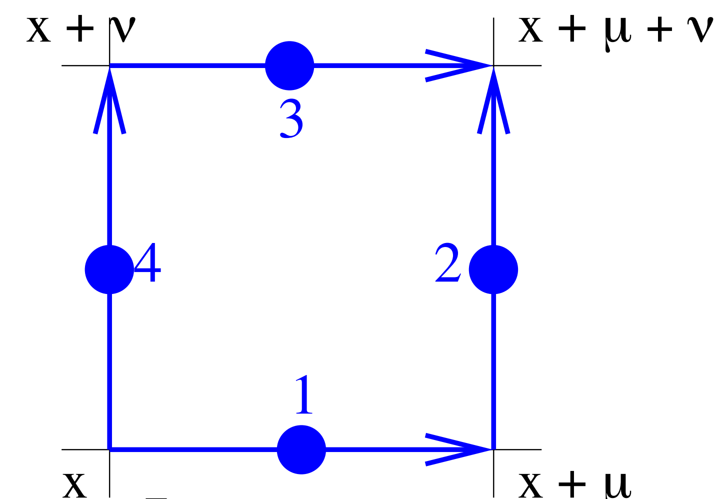
- Integration measure over links: $\int \prod_{x,\mu} dU_\mu(x)$, uniform & normalized "Haar measure"

In continuum limit (see next), all choices of action are equivalent

Continuum limit of Wilson action

- $U_{P_{\mu\nu}}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x), \qquad U_\mu(x) = \exp(i\theta_\mu(x)) \ \in U(1)$

- Introduce *gauge field* $A_\mu(x)$ such that $U_\mu(x) = \exp\left[ig\int_x^{x+\hat{\mu}} dx' A_\mu(x')\right]$



$$a \to 0: \quad U_\mu(x) \approx \exp[igaA_\mu(x+\hat{\mu}/2)]$$

$$U_{P_{\mu\nu}}(x) \approx \exp\left[iga\left(A_\mu(x+\hat{\mu}/2) + A_\nu(x+\hat{\mu}+\hat{\nu}/2)\right.\right.$$
$$\left.\left. - A_\mu(x+\hat{\mu}/2+\hat{\nu}) - A_\nu(x+\hat{\nu}/2)\right)\right]$$

$$\approx \exp\left[-iga^2 \underbrace{(\partial_\nu A_\mu(x+\hat{\mu}/2+\hat{\nu}/2) - \partial_\mu A_\nu(x+\hat{\mu}/2+\hat{\nu}/2))}_{F_{\mu\nu}}\right]$$

$$\mathrm{Tr}U_{P_{\mu\nu}} + \mathrm{Tr}U_{P_{\mu\nu}}^{-1} \approx \exp(-iga^2F_{\mu\nu}) + \exp(+iga^2F_{\mu\nu}) \approx 2 - g^2a^4F_{\mu\nu}^2$$

- Introduce $\beta \equiv \dfrac{1}{g^2}: \quad S_W = \beta \sum_{x,\mu\nu}\left(1 - \mathrm{ReTr}U_{P_{\mu\nu}}(x)\right) \underset{a\to 0}{=} \int d^4x \, \frac{1}{2}F_{\mu\nu}^2(x)$

Other loop shapes: same when $a \to 0$, except $\varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}(U_{P_{\mu\nu}}U_{P_{\rho\sigma}}) \to F\tilde{F}$, $CP-$violating

Continuum limit of Wilson action: $U(1)$ and other groups

- $U(1)$: $\boxed{\beta \equiv \frac{1}{g^2} : \quad S_W = \beta \sum_{x,\mu\nu} \left(1 - \mathrm{ReTr} U_{P_{\mu\nu}}(x)\right)} \underset{a \to 0}{=} \int d^4x \, \frac{1}{2} F_{\mu\nu}^2(x)$

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \longrightarrow \int d^4x \, \frac{1}{2}(E^2 + B^2)$$ EM energy

- Other Lie groups: $U_\mu(x) = \exp\left[ig \int_x^{x+\hat{\mu}} dx' \, \underbrace{A_\mu^k(x')}_{\in \mathcal{R}} \underbrace{\tau^k}_{N \times N}\right]$, τ^k generators, $[\tau^k, \tau^l] = f^{klm} \tau^m$

$$\boxed{\beta \equiv \frac{2N}{g^2} : \quad S_W = \beta \sum_{x,\mu\nu} \left(1 - \frac{1}{N} \mathrm{ReTr} U_{P_{\mu\nu}}(x)\right)} \underset{a \to 0}{=} \int d^4x \, \frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x)$$ YM action

Requiring local gauge symmetry leads *uniquely* to EM/YM action

Continuum limit of lattice $SU(N)$ Yang-Mills: asymptotic freedom

- $\beta-$ function: $\beta(g) \equiv \frac{dg}{d\log\mu} = -b_0 g^3 - b_1 g^5 + \cdots\ ,\ b_0 = \frac{1}{16\pi^2}\frac{11N}{3} > 0,\ b_1 = \left(\frac{1}{16\pi^2}\right)^2 \frac{34N^2}{3}$

  b_0 (1-loop), b_1 (2-loop) universal

- Integrate (1-loop): $\frac{1}{b_0}\frac{dg}{g^3} = -d\log\mu \quad \to \quad \mu \sim \exp\left(\frac{1}{2b_0}\frac{1}{g^2}\right)$

  $g \to 0 \quad \Longleftrightarrow \quad \mu \to +\infty\ ,$ asymptotic freedom

- Lattice: UV scale $\mu \sim a^{-1}$, so $a(g) \sim \exp\left(-\frac{1}{2b_0}\frac{1}{g^2}\right)$, ie. $g \to 0 \quad \Longleftrightarrow \quad a \to 0$

  Continuum limit is approached as $g \to 0 \quad (\beta = \frac{2N}{g^2} \to \infty)$ (critical point)

  Scaling law is different from 2nd-order critical point $\xi \sim (g - g_c)^{-\nu}$

Detect continuum behaviour by [perturbative] scaling:

$$(m_{\rm phys} a)(\beta) = \text{const.} \times \exp(-\frac{1}{4b_0 N}\beta)$$

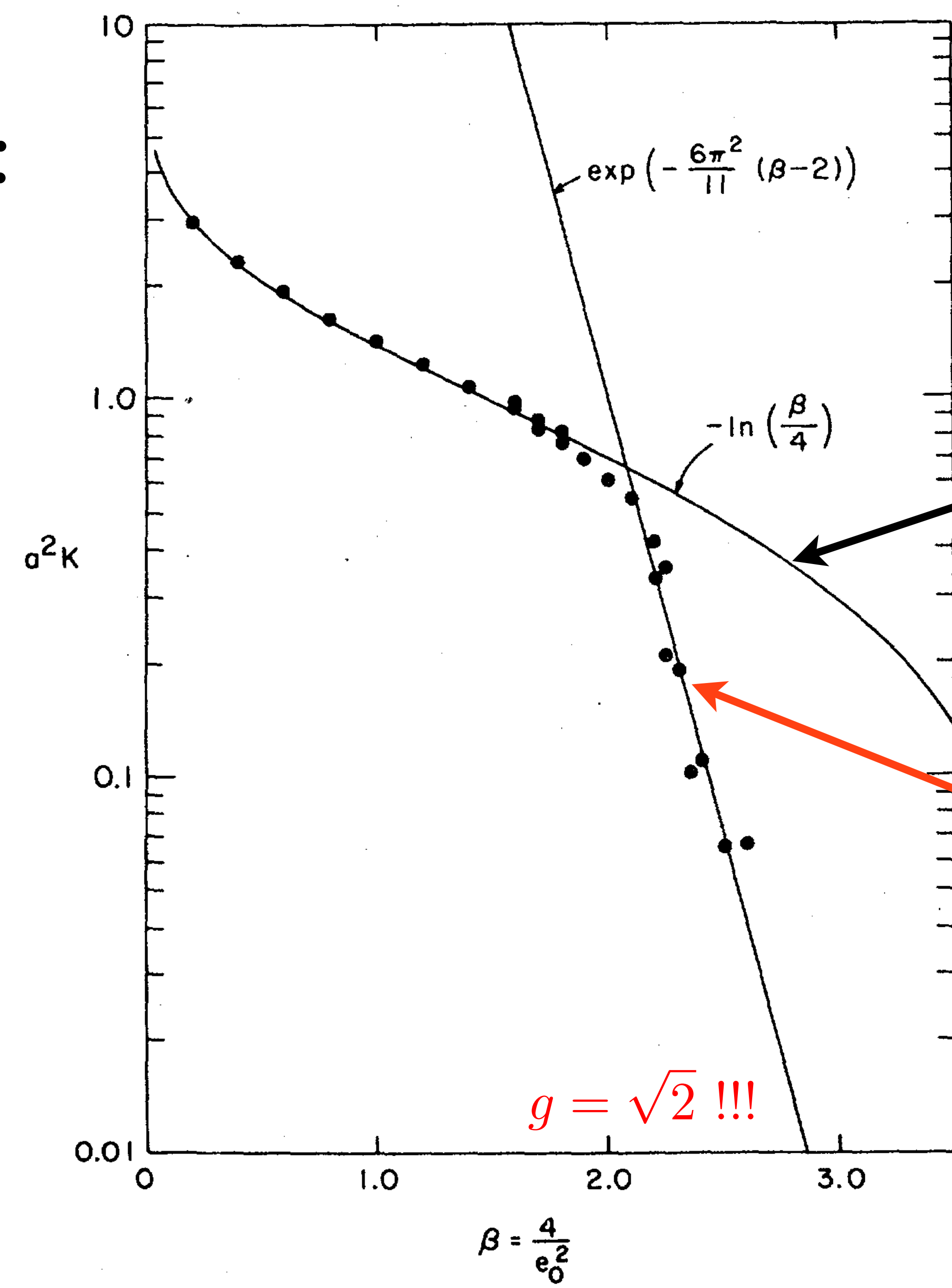
Creutz SU(2): PRD 21 (1980) 2308

[Linear] Confinement:

$$V(r) \underset{r\to\infty}{\sim} Kr$$

K: *string tension*

$$(aV(r)) \sim (a^2K)(r/a)$$



Strong-coupling expansion

1-loop perturbative scaling

FIG. 6. The cutoff squared times the string tension as a function of β. The solid lines are the strong- and weak-coupling limits.

Less stringent requirement: scaling

- Perturbative scaling (1-loop, 2-loop, ...) satisfied accurately for very large β only

  Unnecessarily demanding

- Continuum behaviour is established if "scaling":

$$\frac{(am_i)(\beta)}{(am_{\rm ref})(\beta)} = \text{const.}, \quad i = 1, 2, \cdots$$ as β is increased ($m_{\rm ref} =$, eg, pion)

- Scaling violations polynomial in a :

$$\frac{(am_i)(\beta)}{(am_{\rm ref})(\beta)} = \text{const.} \times (1 + c_1 a + c_2 a^2 + \cdots) \quad \longrightarrow$$ guide $a \to 0$ extrapolation

- "Improved" actions: $c_1 = 0$

Precision groundstate mass/form factor/... available or underway: "1-body" physics under control

Lattice QCD Monte Carlo: sources of errors

- Systematic errors:

  $L \to \infty$, thermodynamic limit

  $a \to 0$, continuum limit

  $m_q \searrow m_{\rm phys}$

Extrapolations guided by analytic ansätze (asymptotic freedom, χPT)

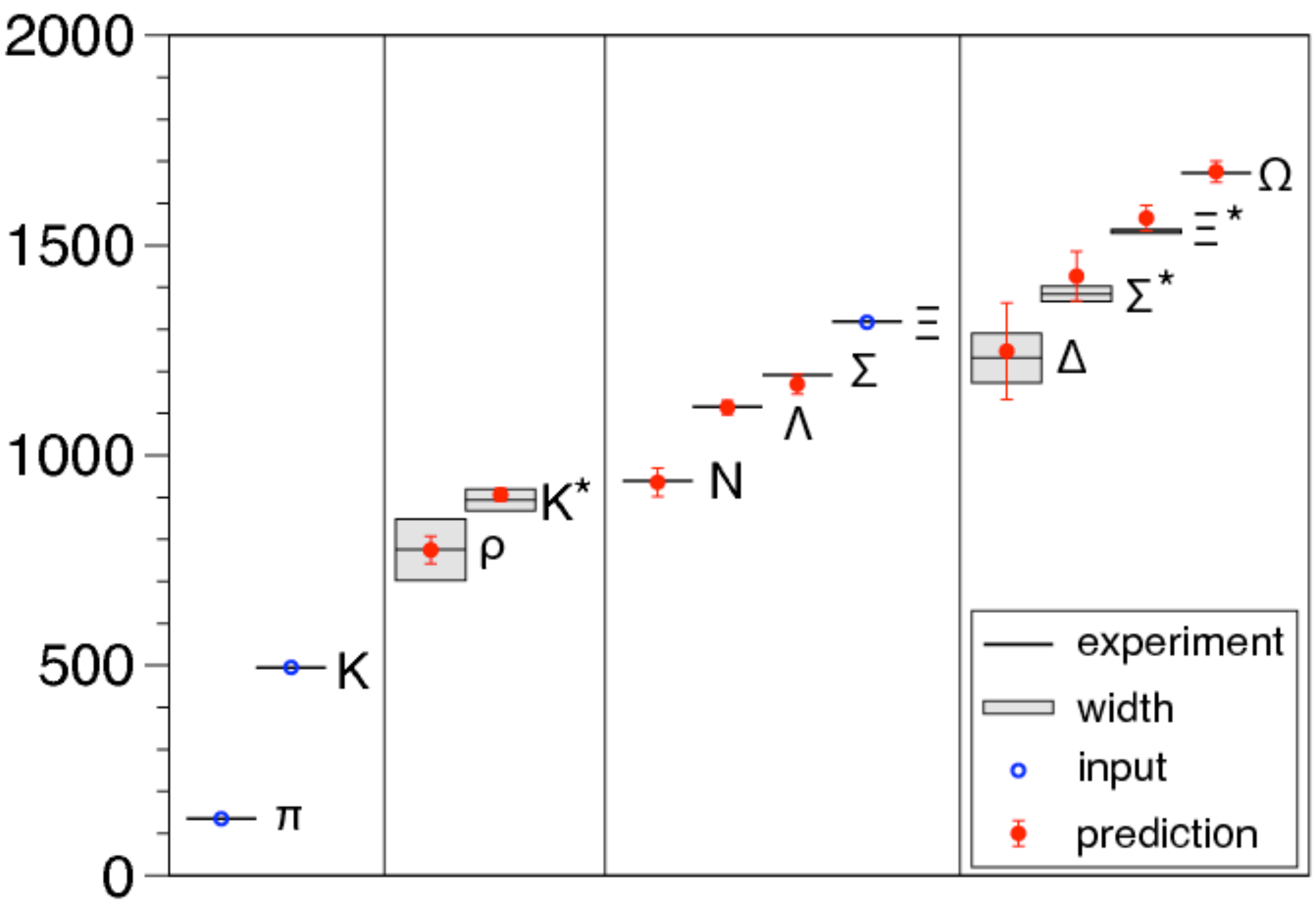
- Statistical (Monte Carlo) errors: $\propto 1/\sqrt{\#\text{configs.}}$

30 years of steady progress since Mike Creutz, 1980:

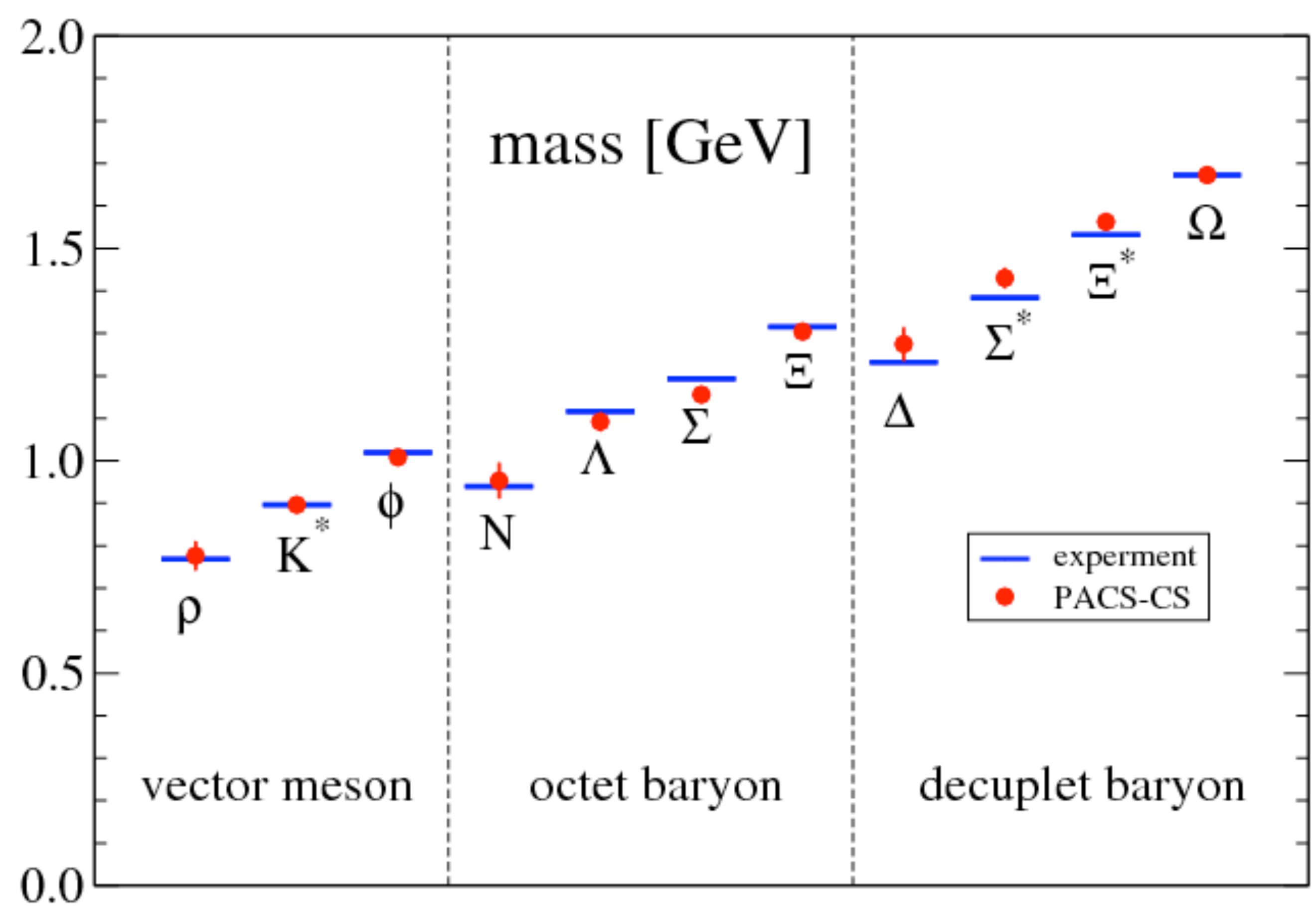
Both errors have been shrinking thanks to hardware + algorithmic progress

$\rightarrow$ Universal tool for *static, equilibrium* properties of QFT

Example: hadron masses



BMW collaboration



PACS-CS collaboration