Outline

- Lattice field theory:
  - Quantum Mechanics with Path Integral (demo)
  - Lattice ϕ^4
  - Scalar QED $\rightarrow$ QCD
- Monte Carlo
- Finite temperature: Y-M deconfinement transition
- Fermions:
  - Continuum symmetries
  - Species doubling
  - Numerical simulation
  - Finite temperature
- Finite chemical potential:
  - Expectations
  - Sign problem
  - Imaginary chemical potential

QFT on the lattice: ϕ^4

- Start from Euclidean Lagrangian density:

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{g_0}{4!}\phi_0^4, \quad S_E = \int d^3x d\tau \; \mathcal{L}_E \underset{T=0}{=} \int d^4x \; \mathcal{L}_E$$

Note: $g_0 \geq 0$ (ie. repulsion), otherwise action unbounded from below

- Same path integral formalism as in QM: $Z = \int \mathcal{D}\phi_0 \; \exp[-S_E(\phi_0(x,\tau))]$

- At each time τ , field configuration $\phi_0(x) : \mathcal{R}^3 \to \mathcal{R}$

Discretize space and time: hypercubic grid, lattice spacing a (could choose $a_\tau \neq a_s$)

$$\int d^4x \to a^4 \sum_x; \qquad \partial_\mu \phi_0(x) \to \frac{\phi_0(x+\hat{\mu}) - \phi_0(x)}{a}$$

- Dimensionless variables: $[\mathcal{L}_\mathcal{E}] \sim \mathrm{mass}^4 \;\Rightarrow\; [\phi_0] \sim \mathrm{mass}, \;\; [g_0] \sim \mathrm{mass}^0$

$$a\phi_0 \equiv \sqrt{2\kappa}\phi; \quad a^2 m_0^2 \equiv \frac{1-2\lambda}{\kappa} - 8; \quad g_0 \equiv \frac{6\lambda}{\kappa^2}$$

$$\longrightarrow \; S_L = \sum_x \left[-2\kappa \sum_\mu \phi(x)\phi(x+\hat{\mu}) + \phi(x)^2 + \lambda(\phi(x)^2 - 1)^2 - \lambda \right]$$

Lattice action: $S_L = \sum_x \left[-2\kappa \sum_\mu \phi(x)\phi(x+\hat{\mu}) + \phi(x)^2 + \lambda(\phi(x)^2 - 1)^2 - \lambda \right]$

- κ "hopping parameter"; $\lambda \to +\infty$ Ising limit ($\phi(x) = \pm 1$)

- Two parameters as before: $(m_0, g_0) \leftrightarrow (\kappa, \lambda)$

  $a(\kappa, \lambda)$ determined by *dynamics* (unlike QM)

- Continuum limit $a \to 0$? Physics is defined by [measurable] Green's functions

$$\langle \phi(x)\phi(0) \rangle - \langle \phi \rangle^2 \sim \exp(-\mathbf{m}_{\text{phys}}|x|) \equiv \exp(-|x|/\xi),$$ ξ correlation length

$$\sim \exp(-(am_{\text{phys}}) \times |x/a|); \quad \frac{\xi}{a} = \frac{1}{am_{\text{phys}}}$$

Fix physics (m_{phys} fixed). Then $a \to 0 \implies \frac{\xi}{a} \to \infty$, ie. 2nd order phase transition

Continuum limit coincides with 2nd-order (or higher) phase transition

⟶ (first- versus second-order transition, Ehrenfest, etc..)

Continuum limit coincides with 2nd-order (or higher) phase transition

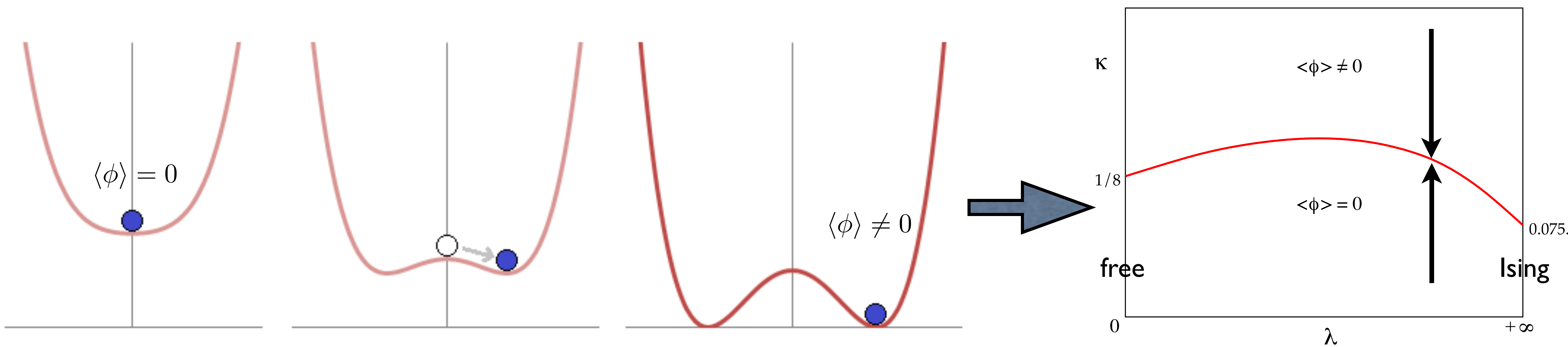
- Universality of critical behaviour: $\tilde{G}(p=0,L,\xi,a)=\tilde{G}(\frac{L}{\xi},\frac{a}{\xi})\underset{a\to 0}{\to}\tilde{G}_{\mathrm{univ}}(\frac{L}{\xi},0)$

  independent of particular lattice discretization

- Must take: thermodynamic limit $L\to\infty$ (first); continuum limit $a\to 0$ (second)

- Second-order P.T. usually associated with *spontaneous breaking of global symmetry*

  Here: Z_2 symmetry $\phi(x)\to -\phi(x)\ \forall x$, S_L unchanged; order parameter $\langle\phi\rangle$



- Note: physical parameters (eg. mass) have *NOTHING* to do with bare parameters (κ,λ)

How to determine physical parameters (m_R, g_R) ?

- Back to continuum QFT: $\tilde{\phi}(k) = \frac{1}{\sqrt{V}} \int d^4x \, e^{-ikx} \phi_0(x), \quad [\tilde{\phi}] = \mathrm{mass}^{-1}$

$$\langle \tilde{\phi}(-k)\tilde{\phi}(k) \rangle \underset{\text{free}}{=} \frac{1}{k^2 + m^2} \underset{g_0 \neq 0}{\longrightarrow} \frac{1}{k^2 + m^2 + \underbrace{\Pi(k^2)}_{c_0 + c_1 k^2 + \cdots}} \approx \frac{1}{\underbrace{1 + c_1}_{\text{rescaling of } \tilde{\phi}}} \times \frac{1}{k^2 + \underbrace{\frac{m^2 + c_0}{1 + c_1}}_{\text{new mass}^2}}$$

$$\tilde{\phi}_R \equiv Z_R^{-1/2} \tilde{\phi}; \quad k = 0$$

$$\longrightarrow \frac{Z_R}{m_R^2} = \langle \tilde{\phi}^*(0)\tilde{\phi}(0) \rangle \underset{x-\text{space}}{=} \langle \int d^4x \, \phi_0(x)\phi_0(0) \rangle \underset{\text{lattice}}{=} 2\kappa a^2 \underbrace{\langle \sum_x \phi(x)\phi(0) \rangle}_{\chi_2}$$

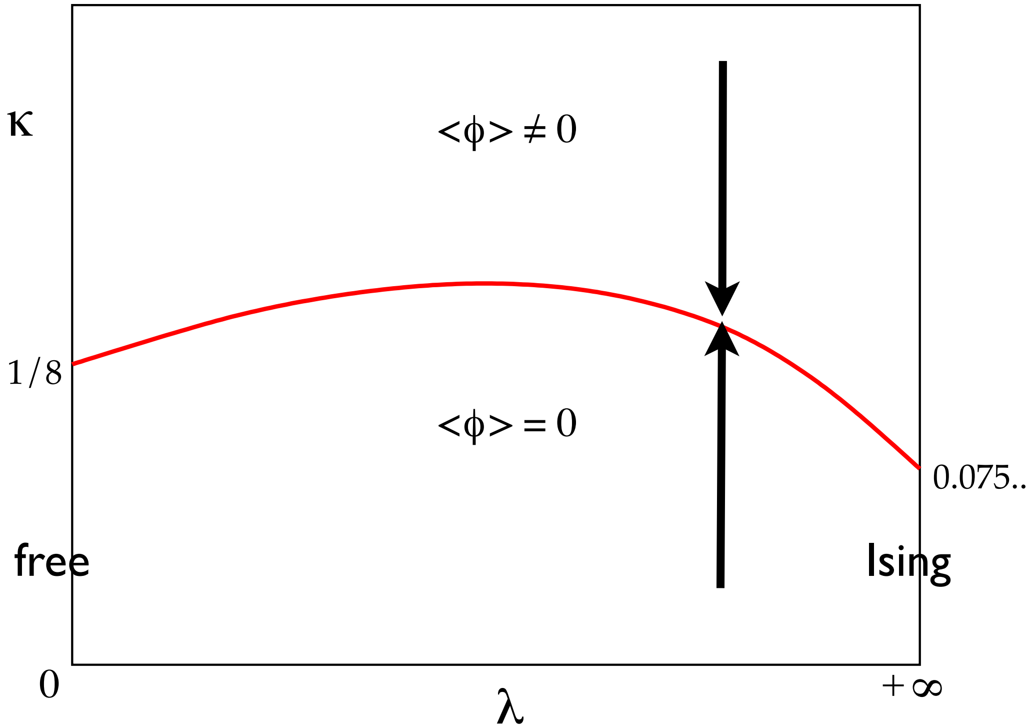
- Similarly: $-\frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k_\mu} \underbrace{\langle \tilde{\phi}(-k)\tilde{\phi}(k) \rangle}_{\frac{Z_R}{k^2 + m_R^2}} \underset{k=0}{=} 2\kappa a^4 \underbrace{\langle \sum_x x^2 \phi(x)\phi(0) \rangle}_{\mu_2}$

- Renormalized coupling g_R from 4-pt fct. $\langle \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) \rangle, k_i = 0 \rightarrow \chi_4 \equiv \langle \sum_{xyz} \phi(x)\phi(y)\phi(z)\phi(0) \rangle$

$$(am_R)^2 = \frac{8\chi_2}{\mu_2}; \quad Z_R = 2\kappa \frac{8\chi_2^2}{\mu_2}; \quad g_R = \frac{64\chi_4}{\mu_2^2}$$

All renormalized quantities measurable by Monte Carlo

Important result: $g_R = 0$ (triviality of ϕ^4)



- Choose *any* λ *(even infinite)* and approach critical line:

$$g_R \to 0$$

- Long-standing theoretical question

- Higgs sector of Standard Model is similar: $\phi \to \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$

- Weakly interacting Higgs sector is just *effective theory*

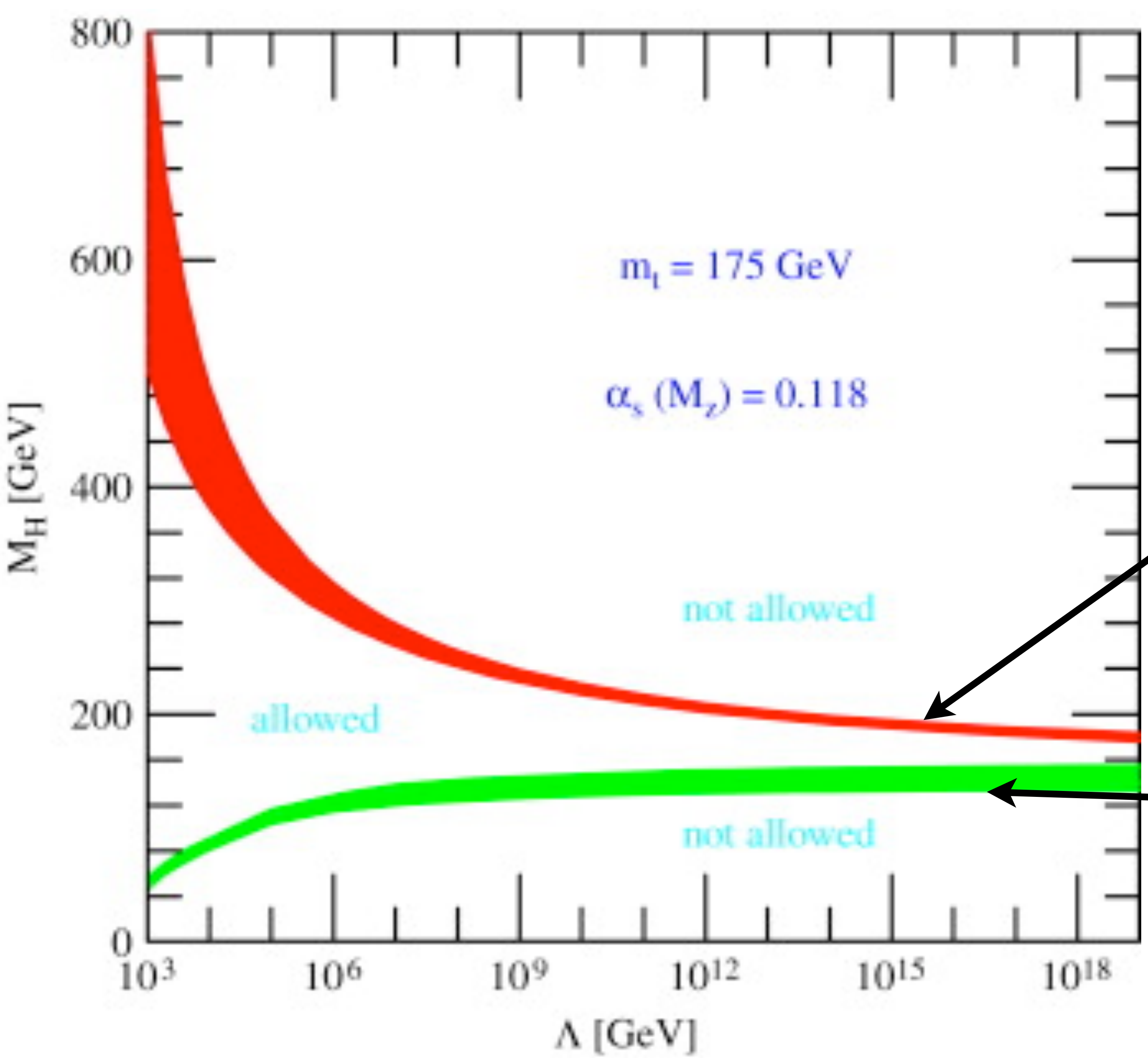
- Tree-level: $V_{\rm tree} = \frac{1}{2} m^2 \phi^2 + \frac{g}{4!}\phi^4, \quad m^2 < 0 \quad \to \quad v = \pm\sqrt{\frac{-6m^2}{g}}$ $\qquad \phi \equiv v + \sigma, \quad V_{\rm tree} \approx -m^2\sigma^2$

$$\frac{\rm mass}{v} = \sqrt{\frac{g}{3}} \quad \text{with} \quad v = 246 \text{ GeV} \qquad \left(\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \underset{M_W = \frac{gv}{2}}{=} \frac{1}{2v^2} \right)$$

Larger Higgs mass $\Longrightarrow$ larger coupling g $\Longrightarrow$ perturbative treatment wrong

Upper bound for Higgs mass from lattice

Higgs mass bounds (upper and lower)



Upper bound from non-perturbative regime

(cf. Landau pole)

Lower bound from vacuum stability

($g(\Lambda)$ becomes negative from top quark)

(cf. 1308.5075)

$m_H = 126$ GeV $\longrightarrow$ $g(\Lambda) = 0$ for Λ near the Planck mass

Higgs mass fine-tuned? "big desert" to Planck mass?