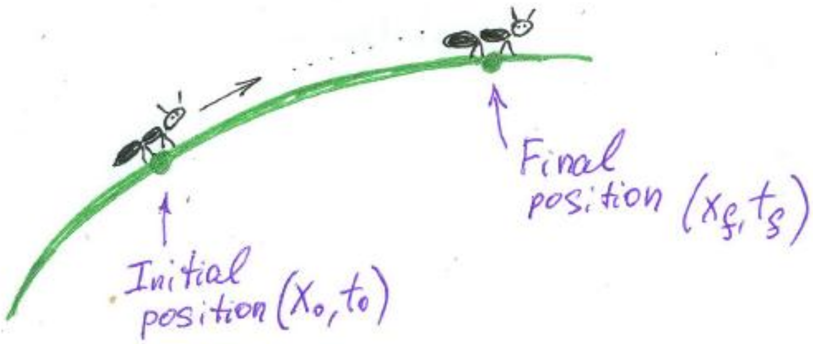
Topological objects and phase transitions in quantum and statistical models

Oleg Pavlovsky (Lomonosov Moscow State University, Russia)

Path Integrals and Probability

Classical Problem:

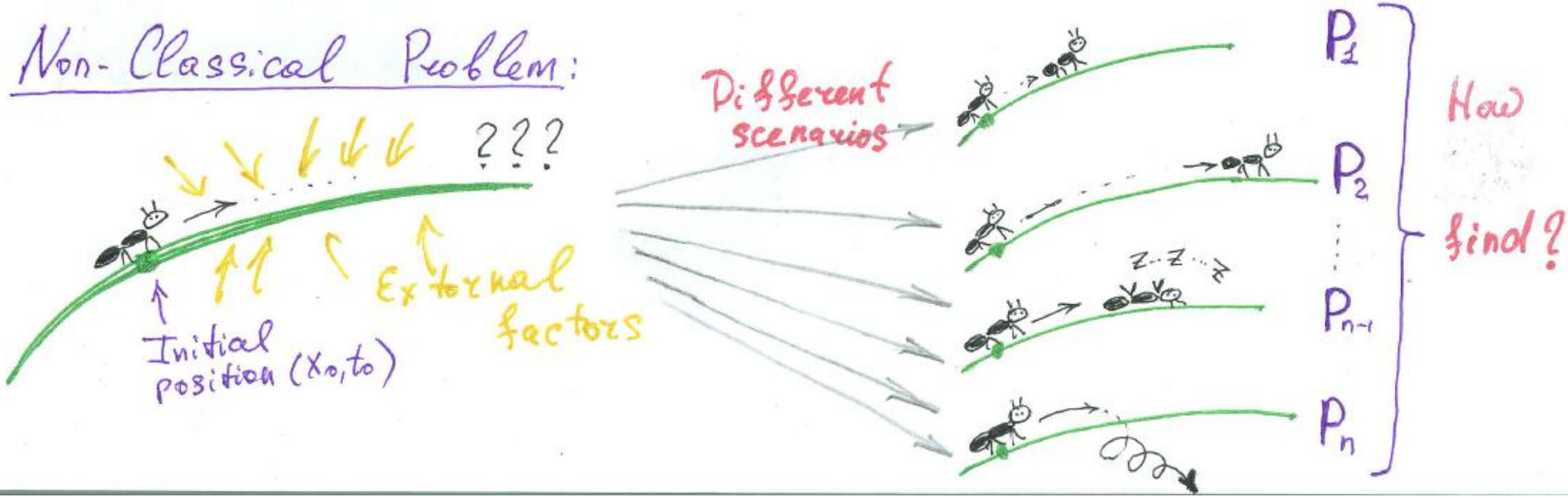


$\exists$ Equation of Motion:

$$\left\{ \begin{array}{l} \ddot{x} = \ldots\ldots \\ x(t_0) = x_0 \end{array} \right| \xrightarrow{\text{Solution}} x(t; x_0, t_0)$$

$$P(x_f, t_f \mid x_0, t_0) = \begin{cases} 0 & \text{if } x_f \neq x(t_f; x_0, t_0) \\ 1 & \text{if } x_f = x(t_f; x_0, t_0) \end{cases}$$

Non-Classical Problem:

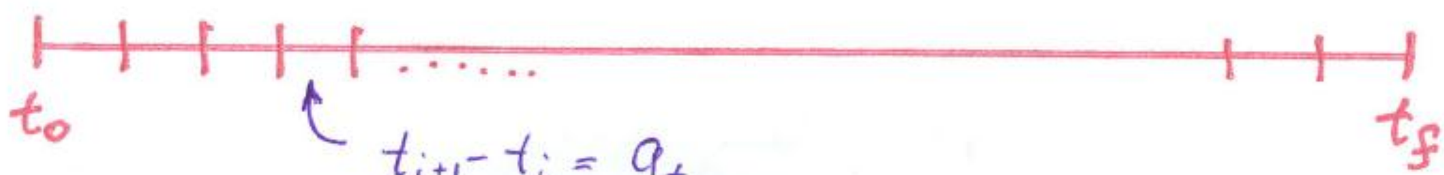


Path Integral : formal introduction

We need $\mathbb{P}(x_f, t_f; x_0, t_0)$.

↳ Simplest way: time Lattice

Let us consider discretisation:



$t_{i+1} - t_i = a_t$

$N_t a_t = t_f - t_0$

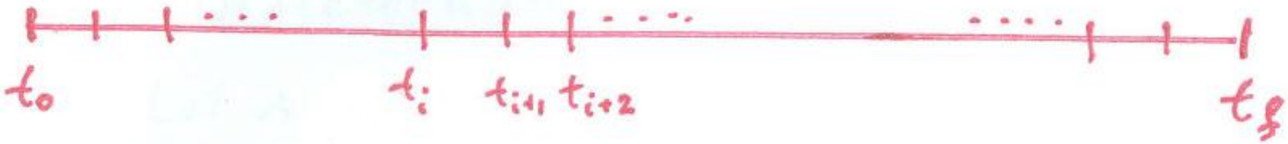
Idea:

1) To work on discret set of times

2) After to take contin. limit for all

$$\lim_{\substack{a_t \to 0,\ N_t \to \infty \\ a N_t = t_f - t_0}}$$

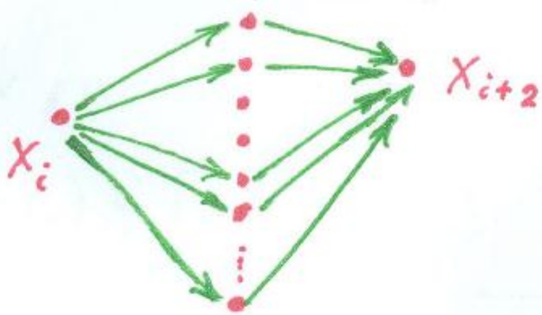
Path Integral: formal introduction



1. Markov process (it must be...)

$$P(x_{i+2}, t_{i+2}; x_i, t_i) = \sum_{x_{i+1}} P(x_{i+2}, t_{i+2}; x_{i+1}\, t_{i+1}) P(x_{i+1}, t_{i+1}, x_i, t_i)$$

This is sum by Path[x_i, x_{i+1}]



So...

$$P(x_f, t_f; x_0, t_0) = \sum_{x_1, x_2 \dots x_{f-1}} P(x_f, t_f, x_{f-1}\, t_{f-1}) P(x_{f-1}, t_{f-1}; x_{f-2}\, t_{f-2}) \cdots P(x_1, t_1; x_0, t_0)$$

Path Integral: formal introduction

2. Differentiability

Let us consider the Hilb. space of vectors:

$|x>$: $\quad \hat{x}|x> = x|x>$, $\quad <x'|x> = \delta(x-x')$

$$\int dx\, |x><x| = 1$$

Evalution operator:

$$P(x_f, t_f; x_0, t_0) = <x_f|\hat{U}(t_f, t_0)|x_0>$$

So, Markov process in terms of $\hat{U}$:

$$\hat{U}(t_f, t_0) = \prod_{0 \le i < f} \hat{U}(t_{i+1}, t_i)$$

$a_t = t_{i+1} - t_i$ is the small parameter!

$$\hat{U}(t+a, t) = \hat{1} - \varepsilon \hat{H}(t) + O(\varepsilon^2) \Rightarrow \frac{\partial}{\partial t}\hat{U}(t,t') = -\hat{H}(t)\hat{U}(t,t')$$

↑ Hamiltonian

Path Integral: Kolmagorov Equation

Path Integral as a Solution of an Equation

$$\frac{\partial}{\partial t_3} \left\{ \hat{\mathcal{U}}(t_3, t_1) = \hat{\mathcal{U}}(t_3, t_2)\, \hat{\mathcal{U}}(t_2, t_1) \right.$$

$\downarrow$ $t_3 \to t_2$

$$\begin{cases} \dfrac{\partial}{\partial t_2} \hat{\mathcal{U}}(t_2, t_1) = -\hat{H}(t_2)\, \hat{\mathcal{U}}(t_2, t_1) \\ \hat{\mathcal{U}}(t, t) = \hat{\mathbb{1}} \end{cases}$$

$\rightarrow$ **Rem. 1** If $\hat{H}(\cancel{t})$:

$$\hat{\mathcal{U}}(t, t') = e^{-(t-t')\hat{H}}$$

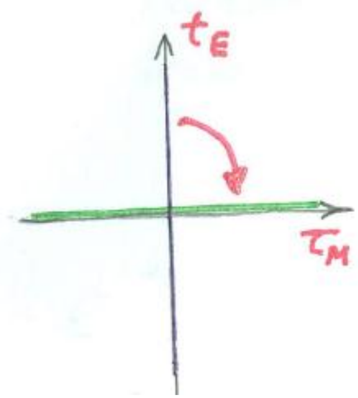
Rem. 2 In QM (Heisenberg Equation)

$$i\hbar \frac{\partial \hat{\mathcal{U}}}{\partial \tau_M} = -\hat{H}\, \hat{\mathcal{U}}$$

Let $\hbar = 1$ But i ?

We work in Euclidian time!

$$t_E = -i\, \tau_M$$



Rem. 3 $\mathcal{Z} = \mathrm{tr}_E\, e^{-\beta \hat{H}}$

$\hookrightarrow$ So,

Euclidian Time t_E $= \beta = 1/T$,

$k_B = 1$ too!

Path Integral: Feynman-Kac formula

Let us consider simple case:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \quad \hat{T}(\hat{p}) = \frac{p^2}{2m} \quad \text{and} \quad V(\hat{x}) = \dots$$

$$P(x_f, t_f; x_0, t_0) = \langle x_f | \hat{U}(t_f, t_0) | x_0 \rangle = \langle x_f | e^{-(t_f - t_0)[T(\hat{p}) + V(\hat{x})]} | x_0 \rangle =$$

time discretisation 

$$= \langle x_f | e^{-a_t[T(\hat{p}) + V(\hat{x})]} \cdot e^{-a_t[T(\hat{p}) + V(\hat{x})]} \cdots\cdots e^{-a_t[T(\hat{p}) + V(\hat{x})]} | x_0 \rangle \approx$$

$[\hat{x}, \hat{p}] \neq 0 \rightarrow$ no basis for diagonalisation $T(\hat{p})$ and $V(\hat{x})$.

↳ one possible way – to use Trotter formula:

$$e^{t(\hat{A}+\hat{B})} = \lim_{n \to \infty} e^{\frac{t}{n}\hat{A}} e^{\frac{t}{n}\hat{B}}$$

↳ $V(\hat{x})$ has no singularity

$$\overset{a_t \to 0}{\approx} \langle x_f | \cdots\cdots e^{-a_t T(\hat{p})} \cdot e^{-a_t V(\hat{x})} \cdots | x_0 \rangle$$

Path Integral: Feynman-Kac formula

$$P(x_f,t_f;x_0,t_0) \overset{a_t\to 0}{=} \langle x_f| \cdots\cdots e^{-a_t T(\hat p)}\, e^{-a_t V(\hat x)}\, e^{-a_t T(\hat p)}\, e^{-a_t V(\hat x)} \cdots |x_0\rangle \ominus$$

$$\mathbb{1} = \frac{1}{2\pi}\int dp_i\, |p_i\rangle\langle p_i| \qquad \mathbb{1} = \int dx_i\, |x_i\rangle\langle x_i|$$

$$\ominus \langle x_f|\cdots\int dx_i \frac{1}{2\pi}\int dp_i\, |p_i\rangle\, e^{-a_t T(p_i)}\, e^{-a_t V(x_i)} \langle p_i|x_i\rangle\langle x_i|\cdots|x_0\rangle \ominus$$

If $T(p) = \frac{p^2}{2m}$:

$$\frac{1}{2\pi}\int dp_i\, e^{-a_t \frac{p_i^2}{2m}} \cdot e^{-i p_i x_{i+1}} \cdot e^{i p_i x_i} =$$

$$= \frac{1}{2\pi}\int dp_i\, e^{-a_t \frac{p_i^2}{2m} - i p_i (x_{i+1}-x_i)} \Rightarrow \text{Gauss Integral}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{m}{a_t}}\, e^{-\frac{1}{2} m a_t \left(\frac{x_{i+1}-x_i}{a_t}\right)^2}$$

Ex. $T(\hat p) = \sqrt{\hat p^2 + m^2}$

$$\overset{a_t\to 0}{\ominus} \left(\frac{m}{2\pi a_t}\right)^{N_t-1} \int \prod_{0\le i<f} \exp\left[-a_t\left(\tfrac{1}{2}m(\nabla x_i)^2 + V(x_i)\right)\right] dx_1\ldots dx_{f-1}$$

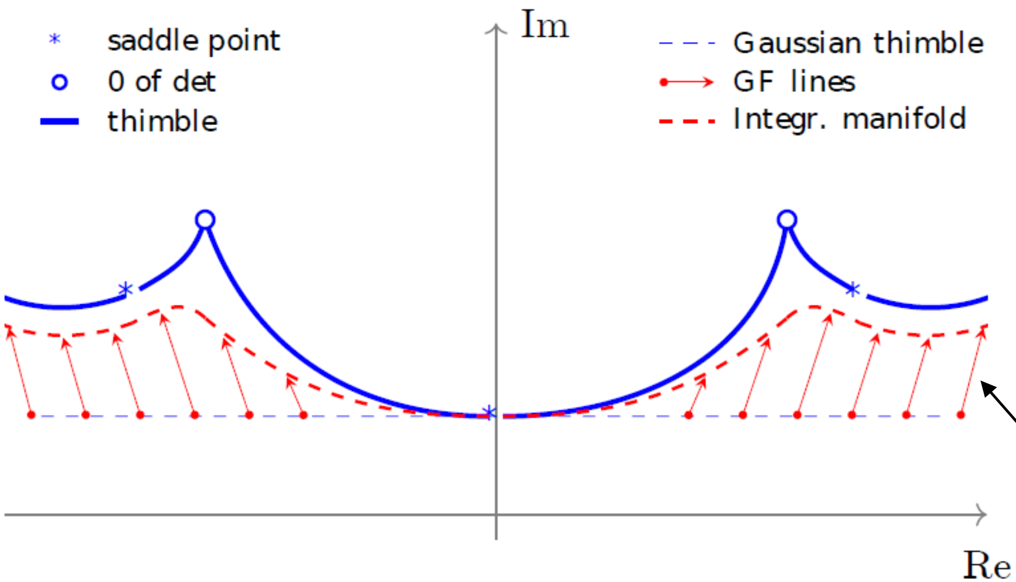
$$\overset{a_t\to 0}{\ominus} \underbrace{N \int dx_1 \ldots dx_{f-1}}_{\int \mathcal{D}x(t)} \exp\Big[-\underbrace{\sum_{0\le i<f} \Delta S_i}_{S_{cl}}\Big] = \int_{\substack{x(t_0)=x_0 \\ x(t_f)=x_f}} \mathcal{D}x(t)\, e^{-S_{cl}[x(t)]}$$

Real Time Path Integrals, Sign Problem and so on…

$$\mathcal{Z}(\beta,\mu,\ldots) = \int_{\mathbb{R}^N} d^N x\, e^{-S(\beta,\mu,\ldots,x)}.$$

$$\mathcal{Z}(\beta,\mu,\ldots) = \sum_\sigma k_\sigma(\beta,\mu,\ldots)\,\mathcal{Z}_\sigma(\beta,\mu,\ldots),$$

$$\mathcal{Z}_\sigma(\beta,\mu,\ldots) = \int_{\mathcal{I}_\sigma(\beta,\mu,\ldots)} d^N x\, e^{-S(\beta,\mu,\ldots,x)},$$



$$\frac{dx}{d\tau} = \overline{\frac{\partial S}{\partial x}},$$

Path Integral: Methods of Calculation

1. Analytical methods
2. Numerical methods: MC

Analytical methods:

1. Direct calculation: Gel'fand-Yaglom method
2. Perturbation expansion
3. Variational methods
4. Quasi-classical method

Path Integral: Quasi-classical approuch

$$P(x_f, t_f; x_0, t_0) = \int\limits_{\substack{x(t_0)=x_0 \\ x(t_f)=x_f}} \mathcal{D}x(\tau)\; e^{-S_{cl}[x(\tau)]}$$

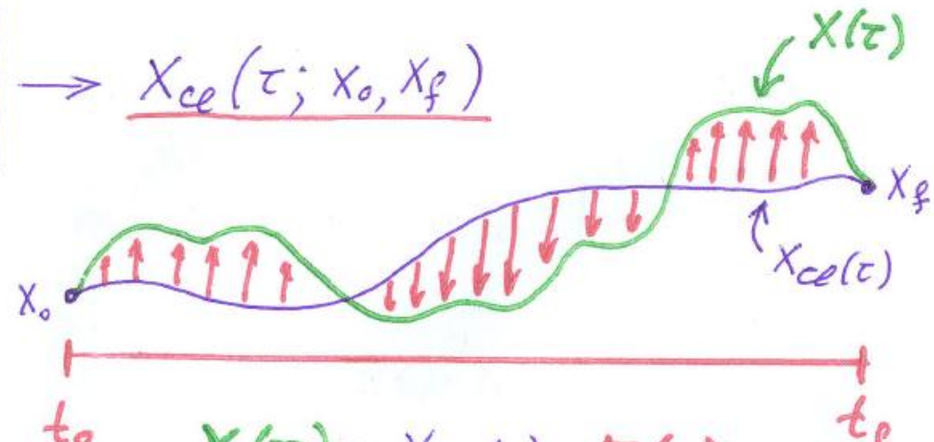
Classical solutions $x_{cl}(\tau)$ have max. stat. weight

Expansion around classical solutions!

↳ Let us consider classical equation of mothion:

$$\begin{cases} \dfrac{\delta S_{cl}}{\delta x(\tau)} = 0 \\ x(t_0) = x_0,\; x(t_f) = x_f \end{cases} \;\Bigg|\; \longrightarrow x_{cl}(\tau; x_0, x_f)$$

↳ Variation $\eta(\tau)$



$$x(\tau) = x_{cl}(\tau) + \eta(\tau)$$

Rem. | Variation work like Deformation!

Quasi-classical Method: simplest example

Free motion (non-relativistic):

$$\hat{H} = \frac{\hat{p}^2}{2m} \Rightarrow S_{cl}[x(\tau)] = \int_{t_0}^{t_f} d\tau \, \frac{1}{2} m \, \dot{x}^2(\tau) \Rightarrow \frac{\delta S}{\delta x(\tau)} = 0$$

$$\Downarrow$$

$$m \ddot{x}_{cl}(\tau) = 0$$

Variation:

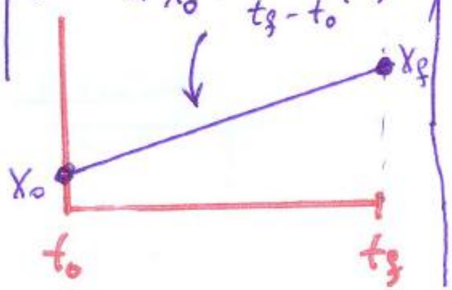
$$S_{cl}[x(\tau) = x_{cl}(\tau) + \tau(\tau)] = \int_{t_0}^{t_f} d\tau \, \frac{1}{2} m \left(\dot{x}_{cl} + \dot{\tau} \right)^2 =$$

$$= \int_{t_0}^{t_f} d\tau \, \frac{1}{2} m \, \dot{x}_{cl}^2 + \int_{t_0}^{t_f} d\tau \, \frac{1}{2} m \, \dot{\tau}^2 +$$

$$+ m \int_{t_0}^{t_f} d\tau \, \dot{x}_{cl} \dot{\tau} = S_{cl}(x_{cl}) + S_{cl}(\tau) + I$$

Equation of motion:

$$\begin{cases} m \ddot{x}_{cl}(\tau) = 0 \\ x_{cl}(t_0) = x_0 \\ x_{cl}(t_f) = x_f \end{cases} \Rightarrow \quad x_{cl}(\tau) = x_0 + \frac{\tau - t_0}{t_f - t_0}(x_f - x_0)$$



$$I = m \int_{t_0}^{t_f} d\tau \, \dot{x}_{cl} \dot{\tau} = m \, \dot{x}_{cl} \tau \Big|_{t_0}^{t_f} - m \int_{t_0}^{t_f} \ddot{x}_{cl} \, \tau \, d\tau$$

First term $= 0$: $\tau(t_0) = \tau(t_f) = 0$; second term: $\ddot{x}_{cl} = 0$ (equat. of M.)

Quasi-classical Method: Free motion case

$$S_{cl}[x(\tau)] = S_{cl}[x_{cl}(\tau)] + S_{cl}[\tau(\tau)] \quad \longleftarrow \quad \underline{\text{Ex.}}$$

Check it for harmonic oscillator

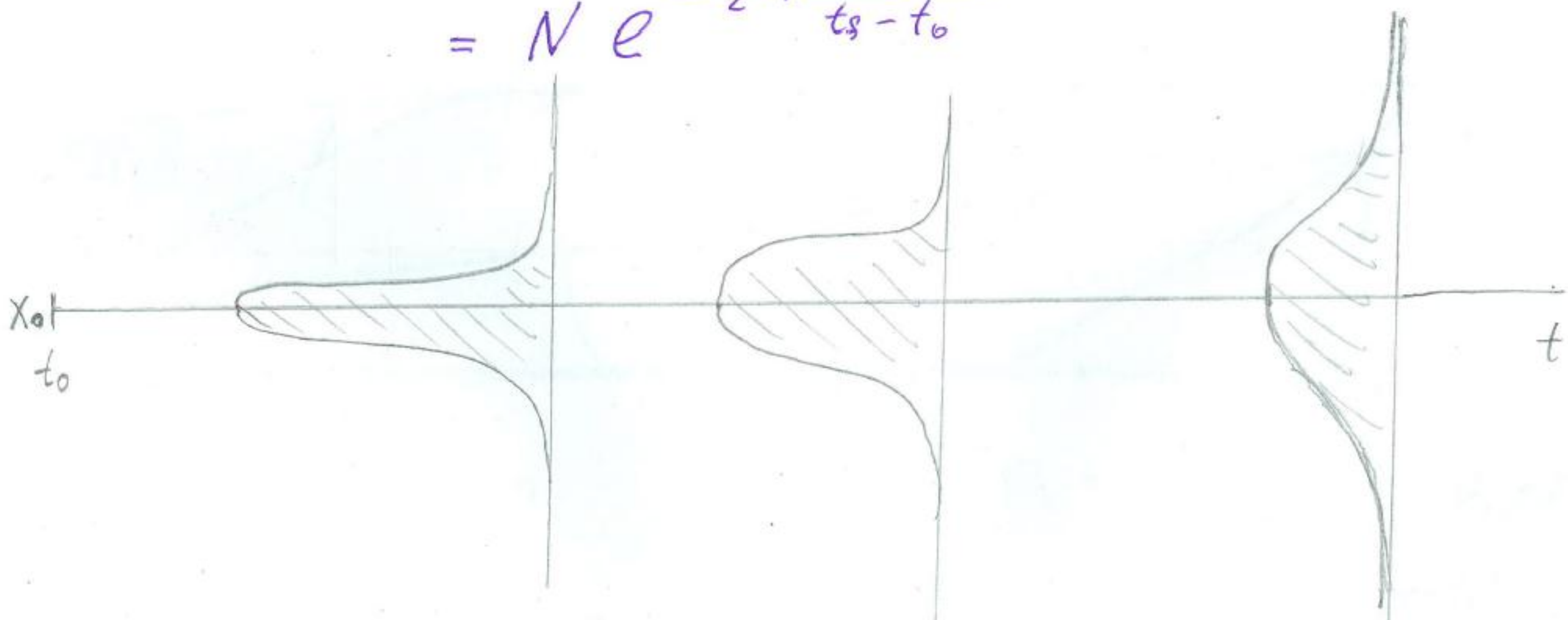
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\omega^2 \hat{x}^2$$

So

$$P(x_f, t_f; x_0, t_0) = \int \mathcal{D}x(\tau)\, e^{-S_{cl}[x(\tau)]} =$$

$$= N(\tau)\, e^{-S_{cl}[x_{cl}(\tau)]} =$$

$$= N\, e^{-\frac{1}{2}m\frac{(x_f - x_0)^2}{t_f - t_0}}$$



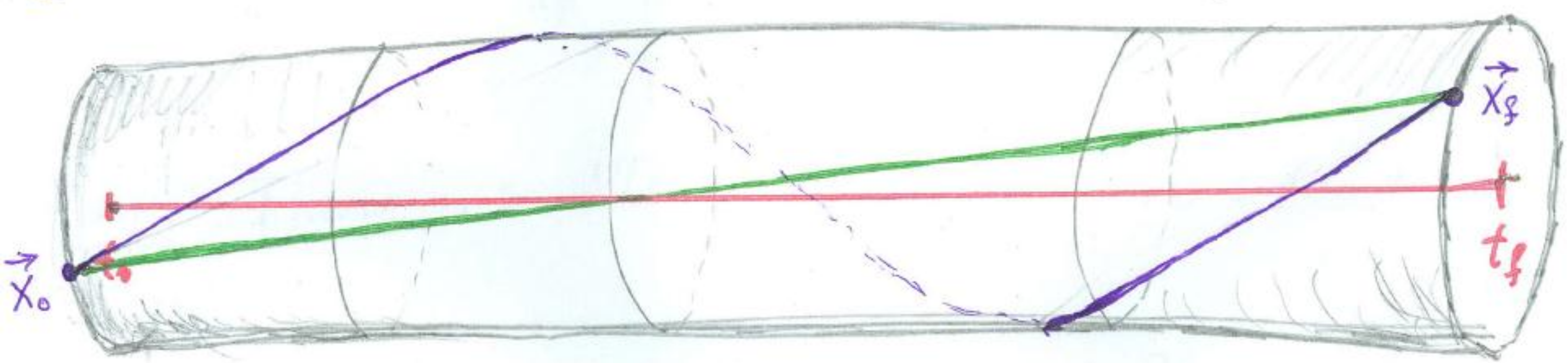
O(2) Symmetric Rigid Rotator: Topological Sectors

Trajectory $\vec{X}(\tau) = (X_1(\tau), X_2(\tau))$

Free motion: $\hat{H} = \frac{1}{2m}\hat{\vec{p}}^{\,2}$

Classical action: $S_{cl} = \int_{t_0}^{t_f} d\tau \left(\frac{1}{2} m \dot{\vec{X}}^2\right) \ominus$

But $X_1(\tau) = R\sin(\theta(\tau))$
$X_2(\tau) = R\cos(\theta(\tau))$ | $\ominus \int_{t_0}^{t_f} d\tau \frac{1}{2} m R^2 \dot{\theta}^2$



Equation of Motion: $\ddot{\theta} = 0 \Rightarrow \theta_{cl} = \theta_0 + \frac{(\tau - t_0)}{t_f - t_0}(\theta_f - \theta_0)$

But!!
There is another one! $\theta_{cl} = \theta_0 + \frac{(\tau - t_0)}{t_f - t_0}(\theta_f - \theta_0 + 2\pi)$

O(2) Rigid Rotator: Topological Sectors

We have sectors:

0 sector: 

$$\theta(\tau) = \theta_0 + \frac{\tau - t_0}{t_f - t_0}\left(\theta_f - \theta_0\right)$$

1 sector: 

$$\theta(\tau) = \theta_0 + \frac{\tau - t_0}{t_f - t_0}\left(\theta_f - \theta_0 + 2\pi\right)$$

2 sector: 

$$\theta(\tau) = \theta_0 + \frac{\tau - t_0}{t_f - t_0}\left(\theta_f - \theta_0 + 4\pi\right)$$

⋮

n sector: 

$$\theta(\tau) = \theta_0 + \frac{\tau - t_0}{t_f - t_0}\left(\theta_f - \theta_0 + 2\pi n\right)$$

⋮

Rem: There is NO smooth $\tau(\tau)$: $\theta \in$ 1 sector deformated to $\theta \in$ 2 sector

⇓

Any sectors give contributions to $P(\vec{x}_f, t_f; \vec{x}_0, t_0)$:

$$P(\bar{x}_f, t_f; \bar{x}_0, t_0) = \sum_{n=-\infty}^{\infty} N_n \exp\left(-\frac{1}{2} m R^2 \frac{(\theta_f - \theta_0 + 2\pi n)^2}{t_f - t_0}\right)$$

Quantization around topological soliton

Let us consider more general case:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \implies S_{ce} = \int_{t_0}^{t_f} d\tau \left(\frac{1}{2} m \dot{x}^2(\tau) + V(x(\tau)) \right)$$

$\Downarrow$

Equation of motion:

$$\begin{cases} m\ddot{x}(\tau) - V'(x(\tau)) = 0 \\ x(t_0) = x_0, \quad x(t_f) = x_f \end{cases} \implies x_{ce}(\tau; x_0, x_f)$$

Variation:

$$x(\tau) = x_{ce}(\tau) + \tau(\tau)$$

$$S_{ce}[x_{ce}(\tau) + \tau(\tau)] = S_{ce}[x_{ce}(\tau)] + \int d\tau_1 \underbrace{\left.\frac{\delta S_{ce}}{\delta x(\tau_1)}\right|_{x = x_{ce}}}_{=0} \tau(\tau_1) + \frac{1}{2!} \int d\tau_1 d\tau_2 \left.\frac{\delta^2 S_{ce}}{\delta x(\tau_1) \delta x(\tau_2)}\right|_{x_{ce}} \tau(\tau_1)\tau(\tau_2)$$

$$+ O(\tau^3) \approx$$

$$\approx S_{ce}[x_{ce}(\tau)] + \frac{1}{2!} \int d\tau_1 d\tau_2 \, \tau(\tau_1) \left.\frac{\delta^2 S_{ce}[x(\tau)]}{\delta x(\tau_1) \delta x(\tau_2)}\right|_{x_{ce}} \tau(\tau_2) \dots$$

Quantization around topological soliton

$$S_{cl}[x_{cl}+\tau] \approx S_{cl}[x_{cl}] + \frac{1}{2}\int d\tau\, \tau(\tau)\left[-\ddot{\tau}(\tau) + V''(x_{cl})\tau(\tau)\right] + \dots$$

Let us consider Sturm-Liouville problem:

$$\begin{cases} -\ddot{\tau}_n(\tau) + V''(x_{cl})\tau_n(\tau) = \varepsilon_n \tau_n(\tau) \\ \tau_n(t_0) = \tau_n(t_f) = 0 \end{cases}$$

→ Steklov Theorem: SL problem generate set of functions $\tau_n(\tau)$:

- ort. basis in L^2! $\{\tau_n(\tau)\}$
- any $\tau(\tau) \in L^2 \Rightarrow \tau(\tau) = \sum_m C_m \tau_m(\tau)$
- $\int d\tau\, \tau_m(\tau)\tau_n(\tau) = \delta_{mn}$

$$S_{cl}[x_{cl}+\tau] \simeq S_{cl}[x_{cl}] + \frac{1}{2}\int d\tau \sum_{nm} C_n \tau_n(\tau)\, \varepsilon_m C_m \tau_m(\tau) = S_{cl}[x_{cl}] + \frac{1}{2}\sum_n \varepsilon_n C_n^2$$

$$P(x_f,t_f;x_0,t_0) = \int \mathcal{D}x(\tau)\, e^{-S_{cl}[x]} = e^{-S_{cl}[x_{cl}]}\, J \int \prod_n dC_n\, e^{-\frac{1}{2}\sum_n \varepsilon_n C_n^2} = N \cdot J \frac{1}{\prod_n \varepsilon_n^{1/2}}\, e^{-S_{cl}(x_{cl})}$$

Smoothness of $\tau(\tau)$, topological sectors, 0-modes ...

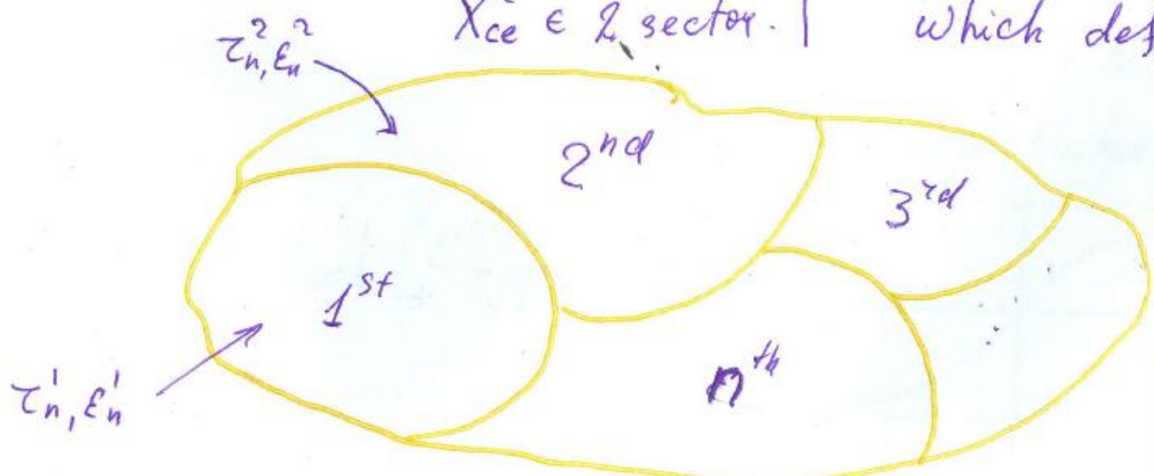
$$P(x_f,t_f;x_0,t_0) = N\frac{1}{\prod_n \varepsilon_n^{1/2}} e^{-S[x_{cl}]} + \dots = N\left(\det\frac{\delta^2 S_{cl}}{\delta x\,\delta x}\right)^{-1/2} e^{-S[x_{cl}]} + \dots$$

Rem 1 If S_{cl} has symmetries $\Rightarrow$ some $\varepsilon_n = 0 \leftarrow$ 0-mode problem

Solution of 0-mode problem – collective coordinate method

Rem 2 Steklov Theorem $\Rightarrow \tau(\tau)$ has No singularities

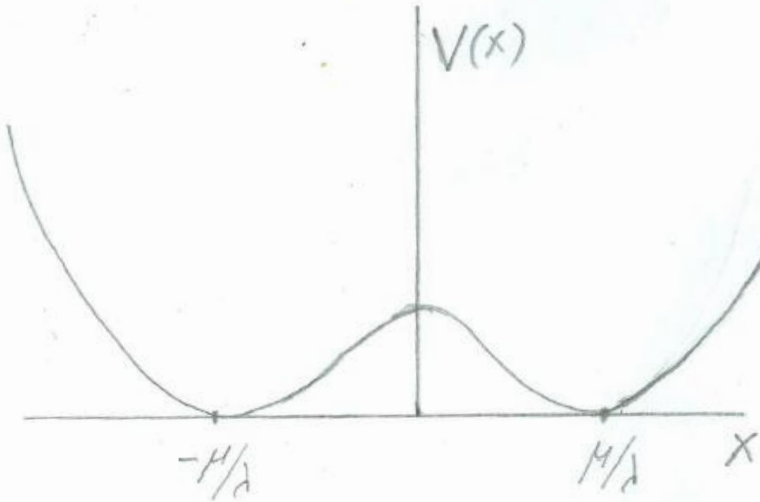
$\hookrightarrow S_0$, $X_{cl}^1 \in$ 1 sector, $X_{cl}^2 \in$ 2 sector. | There are No smooth $\tau(\tau)$ which deform X_{cl}^1 to X_{cl}^2!



$$P(x_f,t_f;x_0,t_0) = \sum_{\text{topology}} N^{top} \det{}^{top} e^{-S[x_{cl}^{top}]} + \dots$$

Topological solitons of φ^4 model

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\lambda}{4}\left(\hat{x}^2 - \frac{\mu^2}{\lambda}\right)^2 \Longrightarrow S_{cl} = \int_{t_0}^{t_f} d\tau \left(\frac{1}{2} m \dot{x}^2(\tau) + \frac{\lambda}{4}\left(x^2(\tau) - \frac{\mu^2}{\lambda}\right)^2 \right)$$



$m = 1$

Equation of motion:

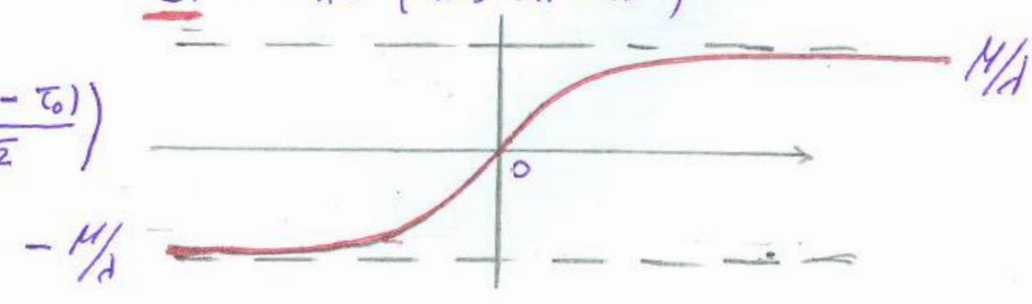
$$\begin{cases} \ddot{x} - \lambda x \left(x^2 - \mu^2/\lambda\right) = 0 \\ x(t_0) = x_0 \\ x(t_f) = x_f \end{cases}$$

Two types of solutions: 1. Classical vacuum:

$$x_{cl}^0(\tau) = \pm \mu/\lambda$$

2. Kinks (instantons)

$$x_{cl}^I(\tau) = \pm \frac{\mu}{\sqrt{\lambda}} \tanh\left(\frac{\mu(\tau - \tau_0)}{\sqrt{2}}\right)$$



Rôle of kinks: qualitative analisis

1. $S_{ce}(X^{0}_{ce}) = 0$

2. $S_{ce}(X^{I}_{ce}) = e^{-2\sqrt{2}\mu^3/3\lambda} > 0$

Partition function:

0-mode

$$Z = tr_E\, e^{-(t_f - t_0)\hat{H}} \approx 2\, e^{-S_{ce}(X^0)} + N(X^I_{ce})\left[\int_{t_0}^{t_f} d\tau_0\right] e^{-S_{ce}(X^I_{ce})} + \cdots$$

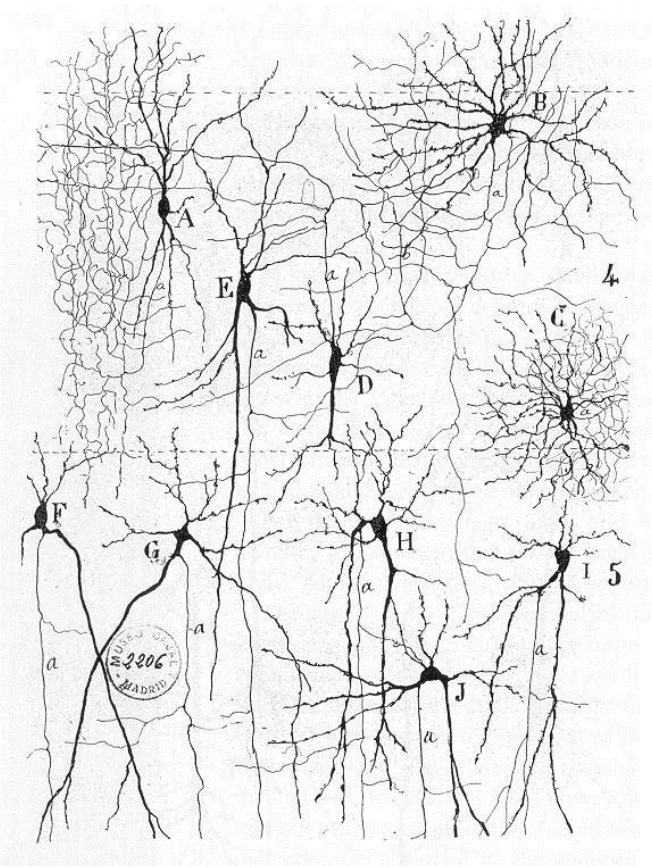
$$\approx 2 + N\,(t_f - t_0)\, e^{-S_{ce}(X^I_{ce})} + \cdots$$

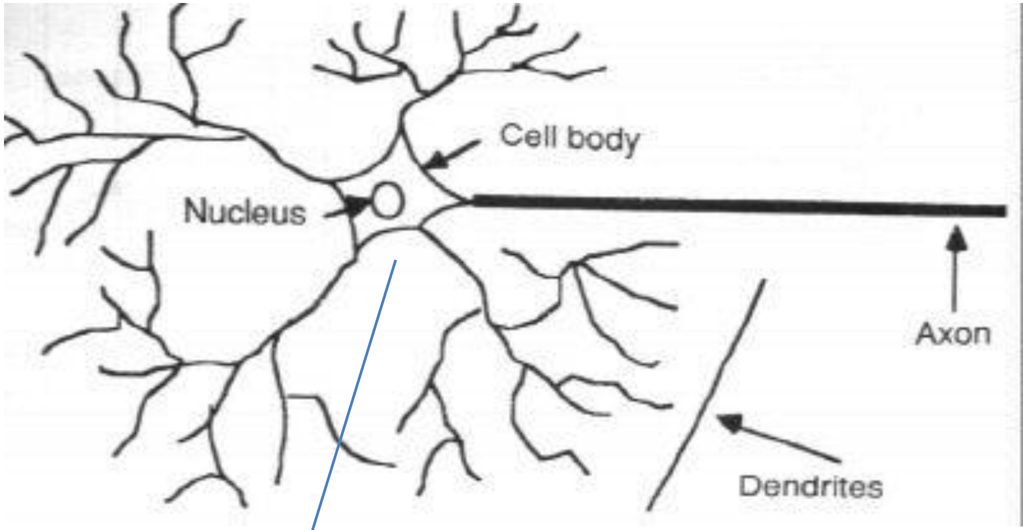
If $(t_f - t_0)\, e^{-S_{ce}(X^I_{ce})} \gg 1$ — Kink regime

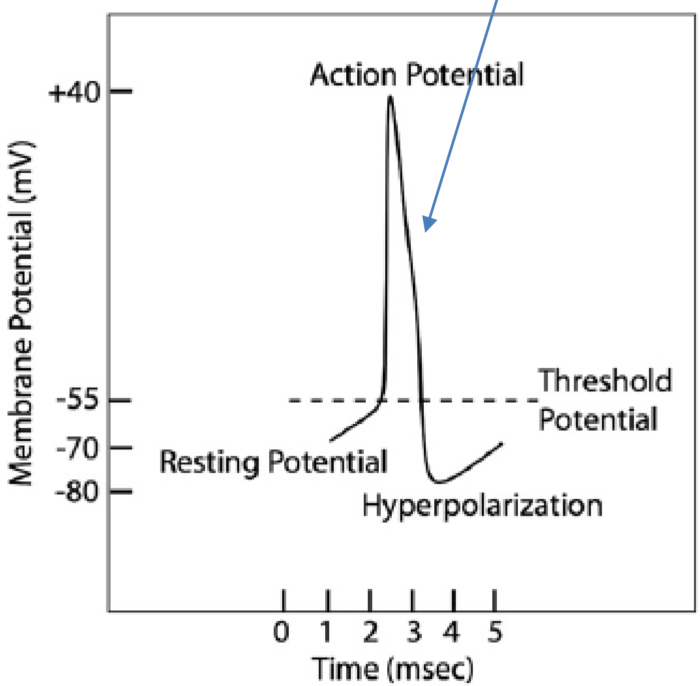
If $(t_f - t_0)\, e^{-S_{ce}(X^I_{ce})} \ll 1$ — Vac. fluctuations $\Rightarrow$ no kinks

•Quantum neural network

•How Neuron Works: Biology

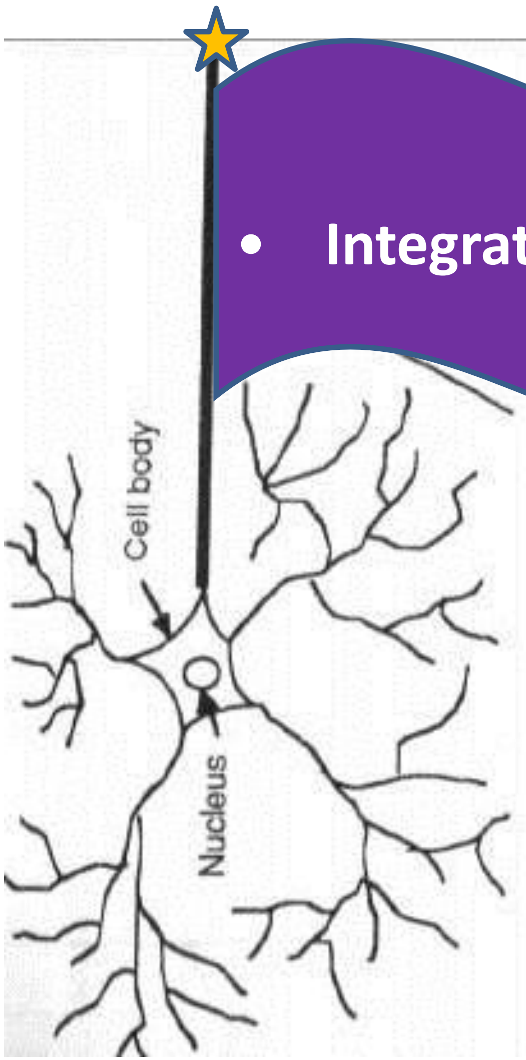


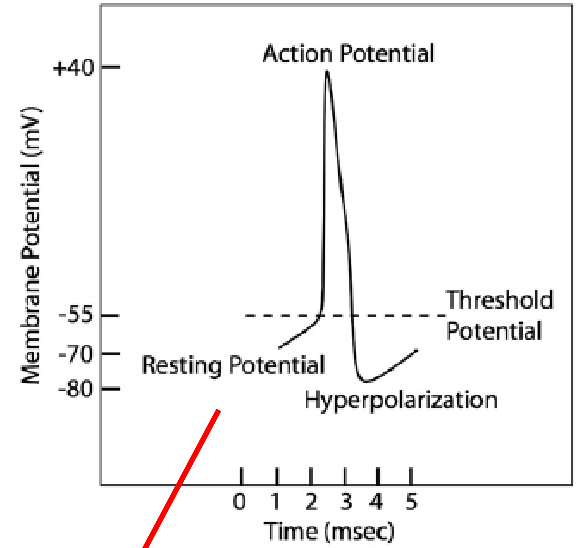




•The models of Biological Neuron

- Integrate-and-fire





$$I(t) = C_{\mathrm{m}} \frac{dV_{\mathrm{m}}}{dt}$$

•**Different for Different Models**

•Artificial Neural Network: main idea

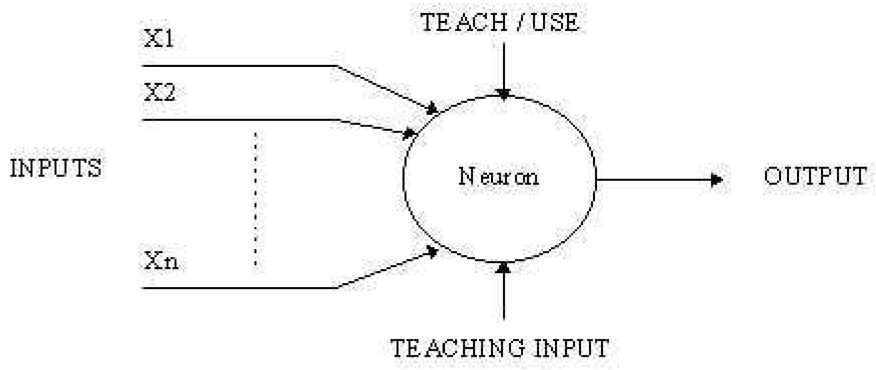
•**Main idea of Artificial Neural Network**:

•**Artificial Neural Network (ANN) is an information system that is inspired**
•**by the biological nervous systems, such as the brain.**

•**The key element of ANN is a large number of highly interconnected processing**
•**elements (neurones) working in unison to solve specific problems.**

•**ANN, like brain, learn by examples or patterns.**

•**Typical ANN problems: pattern recognition, data classification and so on.**



•Neural Network: what we need?

- **Adaptive learning**: An ability to learn how to do tasks based on the data
- given for training or initial experience.
-

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•**High level of Parallelism**

•**Fault Tolerance**

•Artificial Neural Network: main lesson for NeuroBiologists

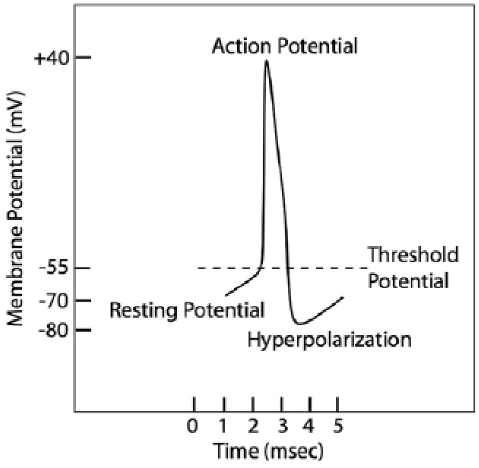
•For modeling the processes in Neural Networks the elements
•of the Network must be simple.

•Main properties we must achieve are:

•Adaptive learning
• Self-Organisation
• High level of Parallelism
• Fault Tolerance

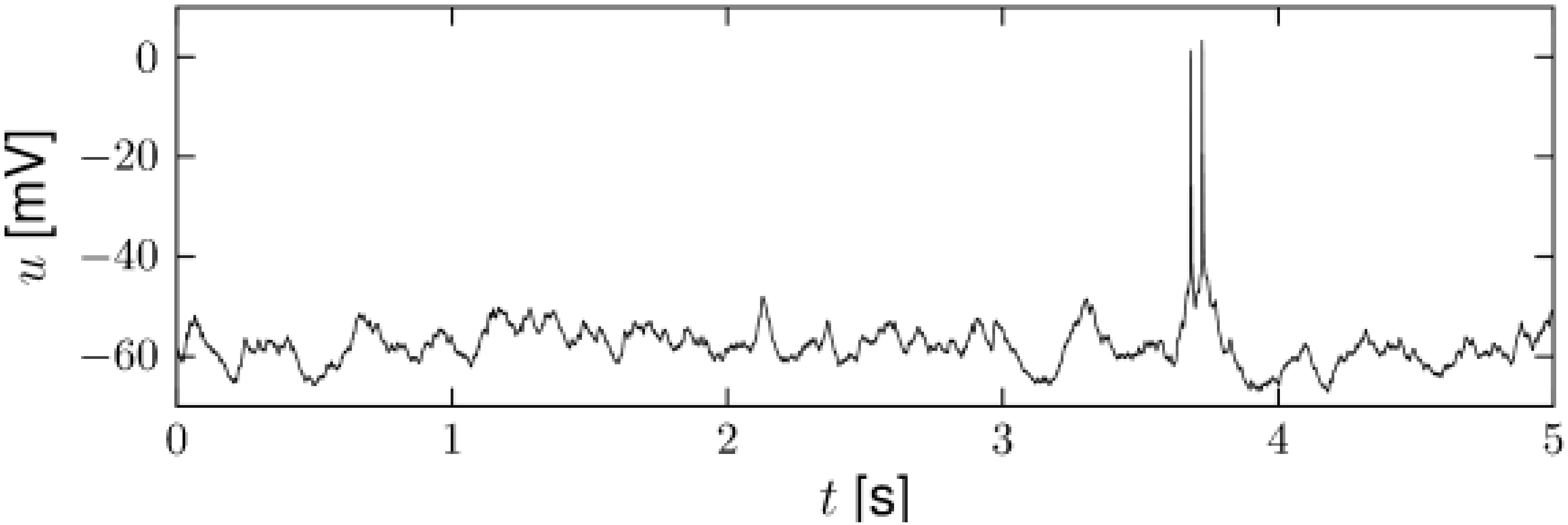
•So…
•
•**Main Question**: **How these properties**
• **can be realized in such Big and Complex**
•**systems as BRAIN?**

•Neuron: classical VS stochastic



•Simplification

•Real neuron potential



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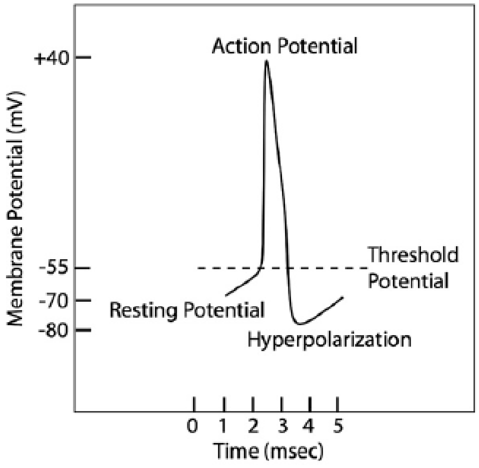

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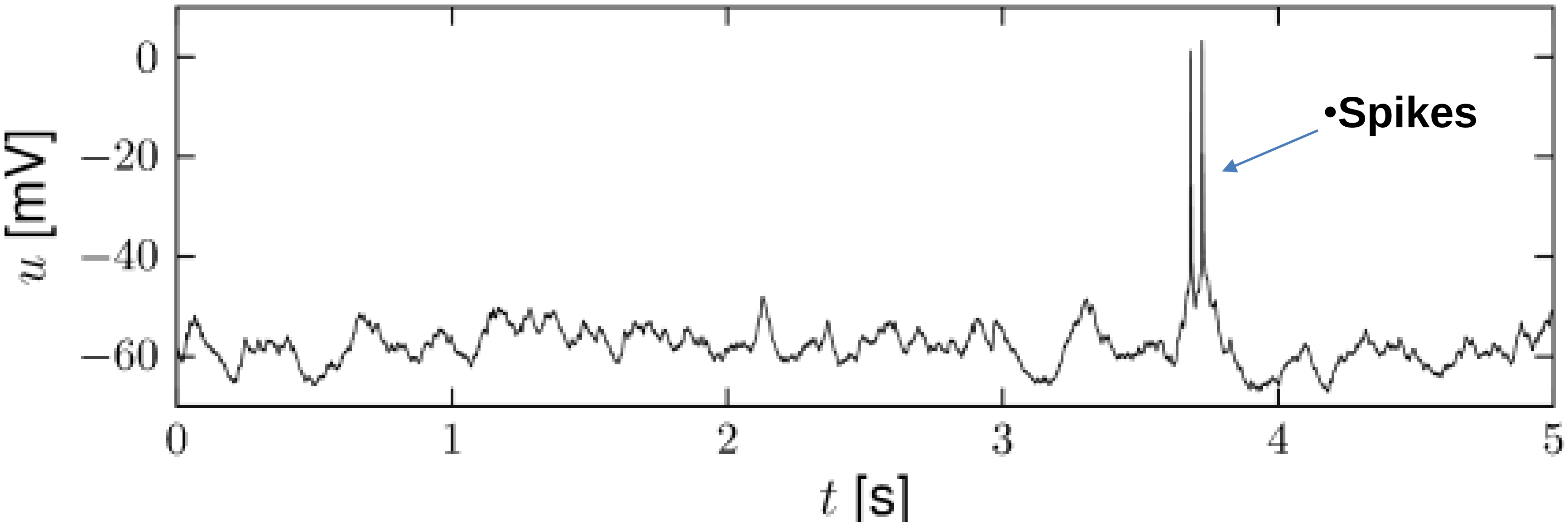

•Spikes

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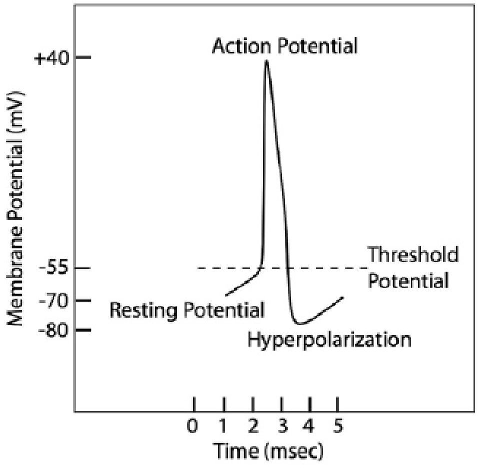

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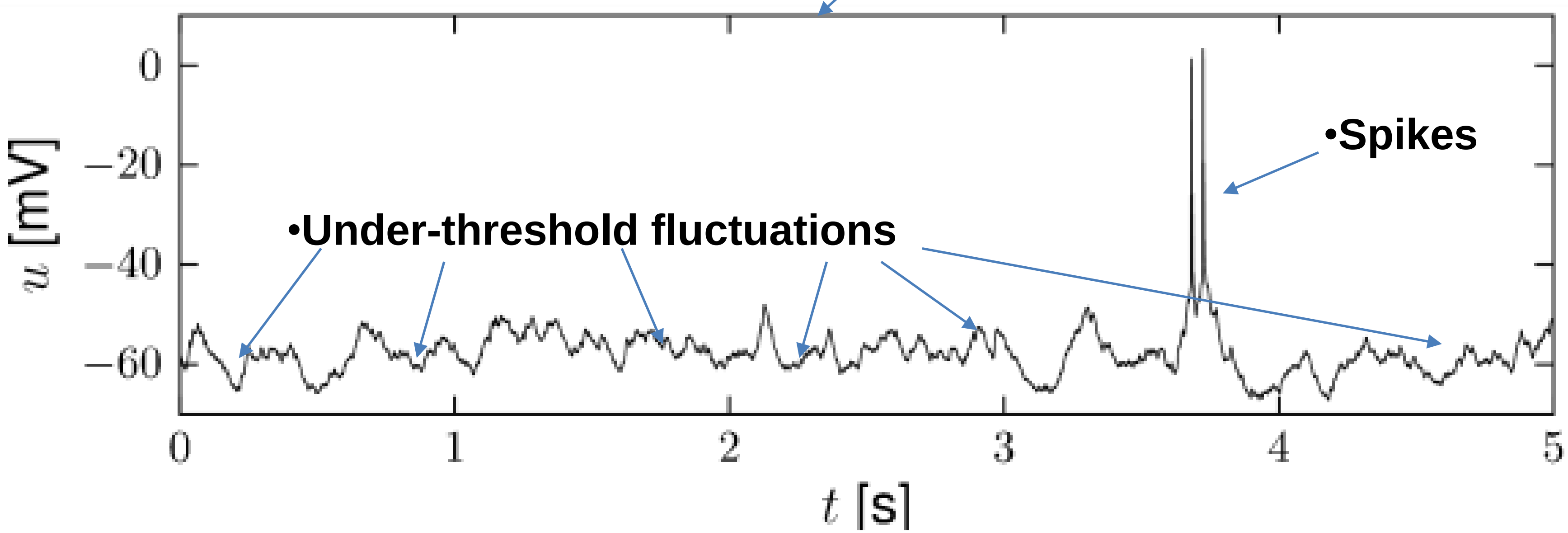
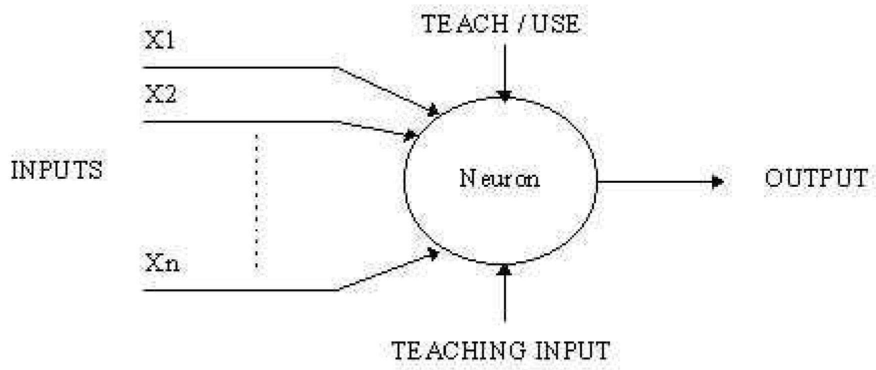
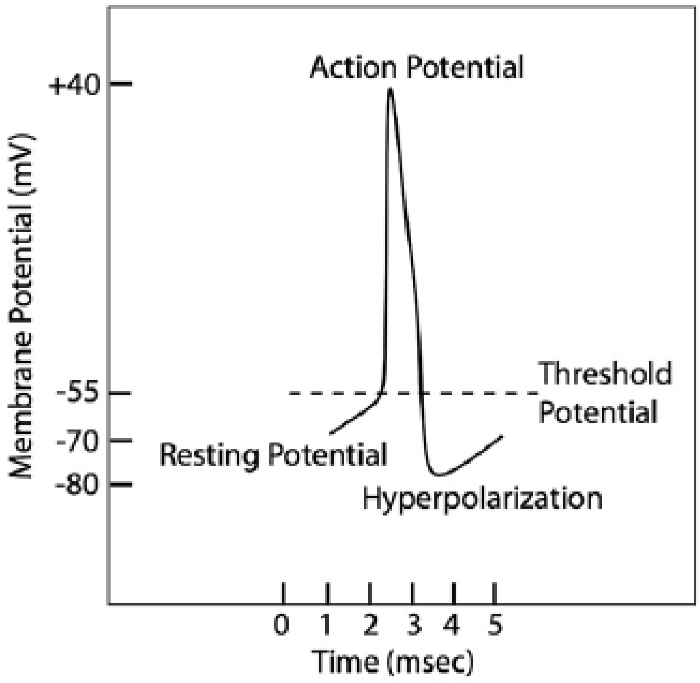


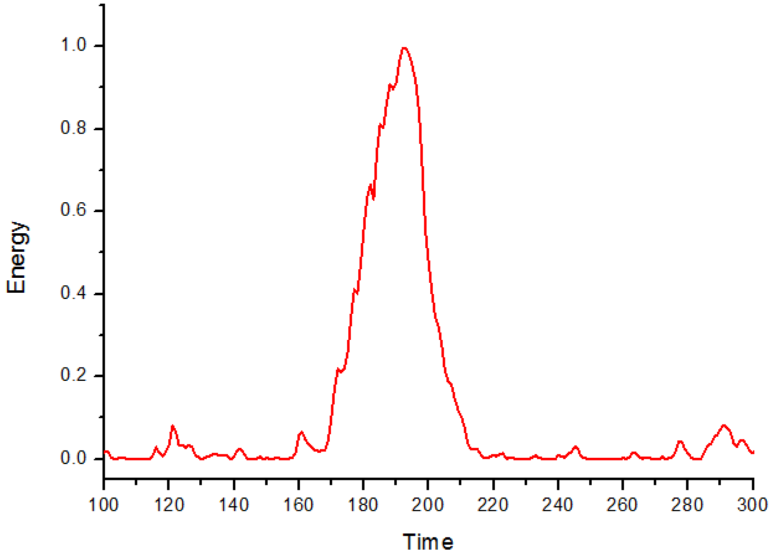
•Spikes

•Under-threshold fluctuations

•Quantum neuron = Q-neuron







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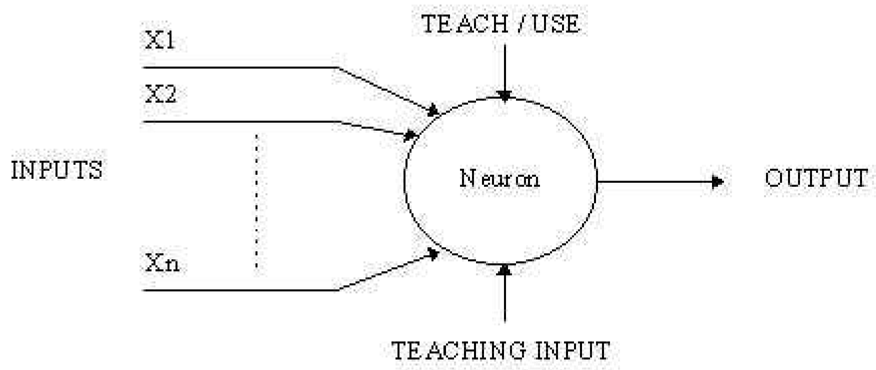
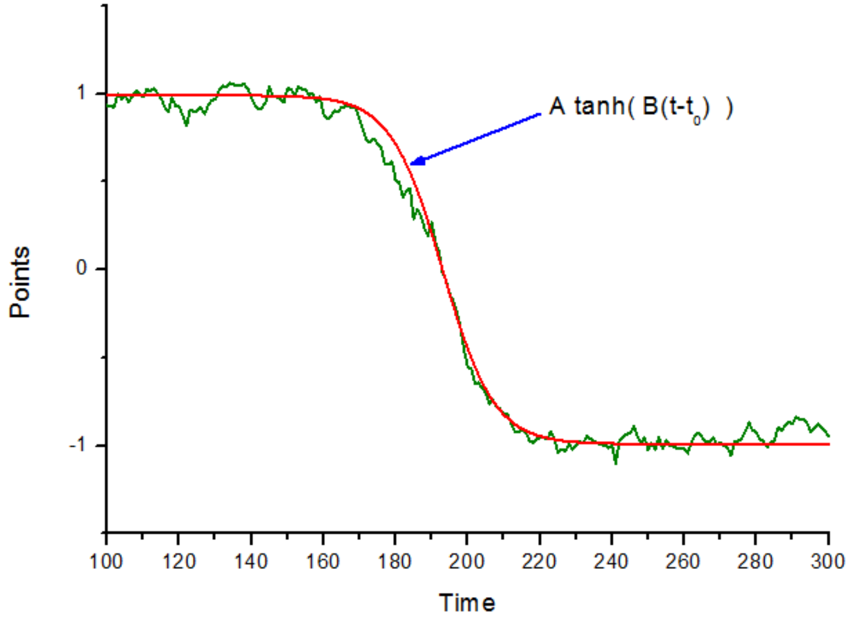
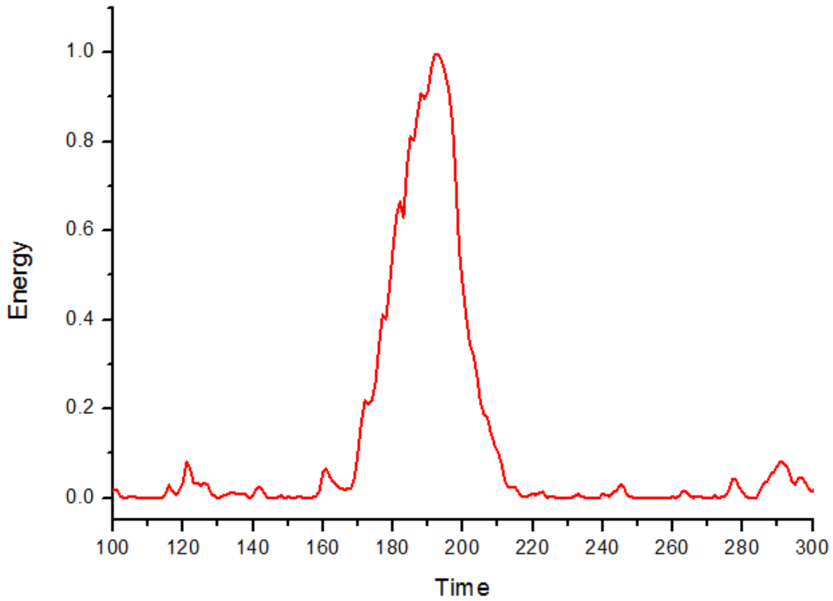
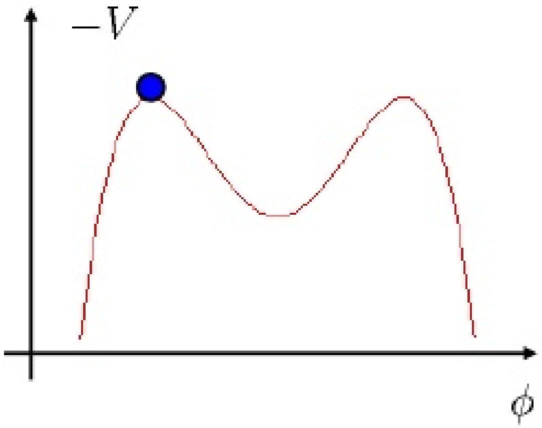


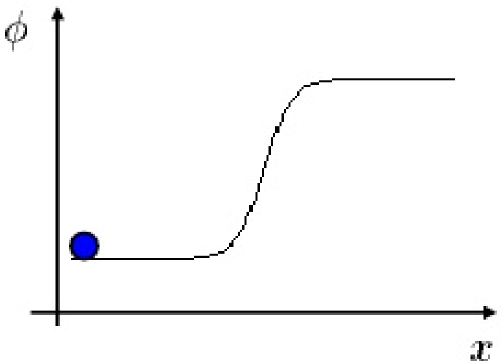




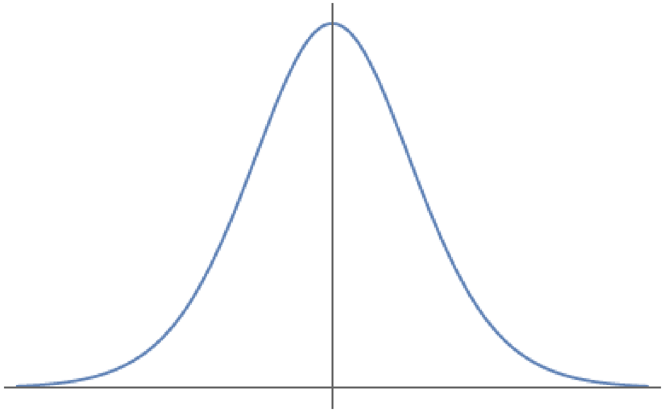
•Quantum neuron

$$\hat{H} = \sum_i \left(\frac{1}{2}\hat{p}_i^2 + V_0(\hat{q}_i) \right) \qquad V_0(q_i) = \frac{\Lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\Lambda}\right)^2.$$

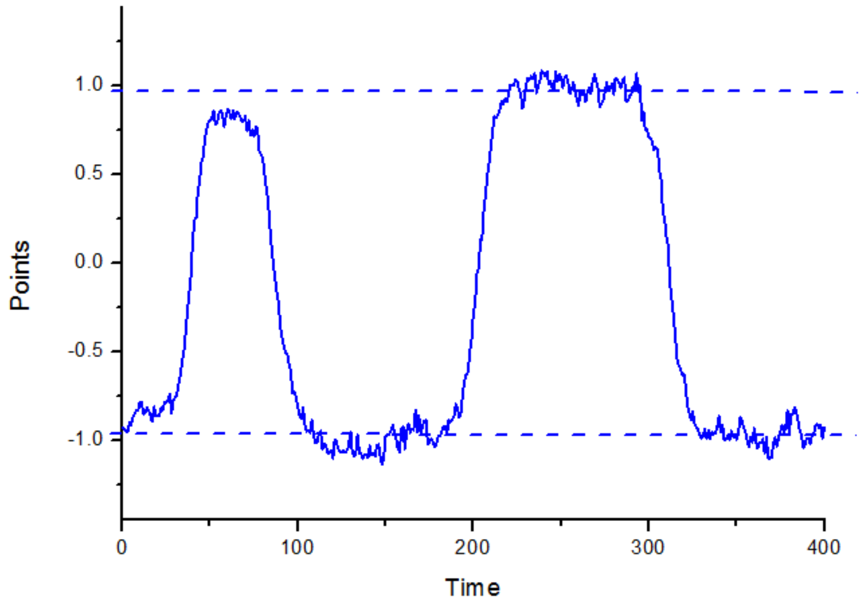


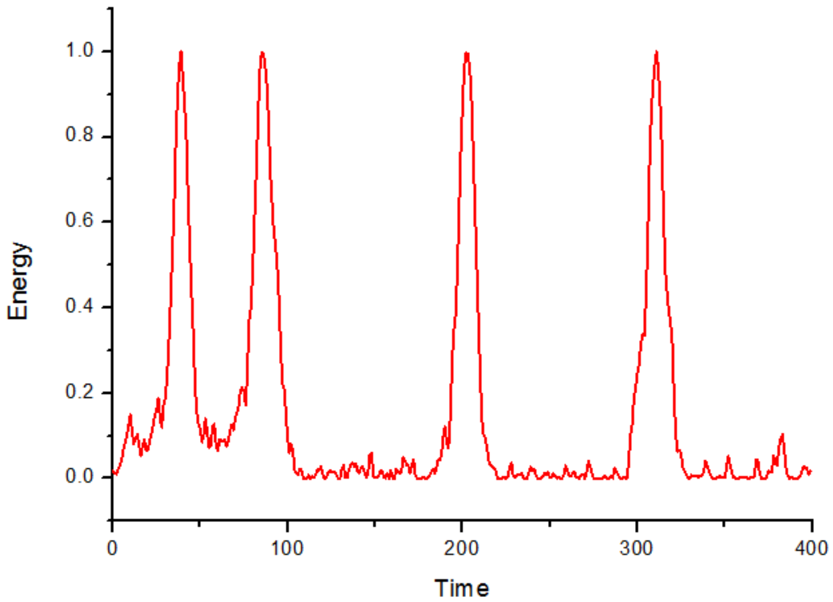


$$\phi(x,t) = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{m}{\sqrt{2}}(x - x_0) \right)$$



•Quantum neuron = Q-neuron





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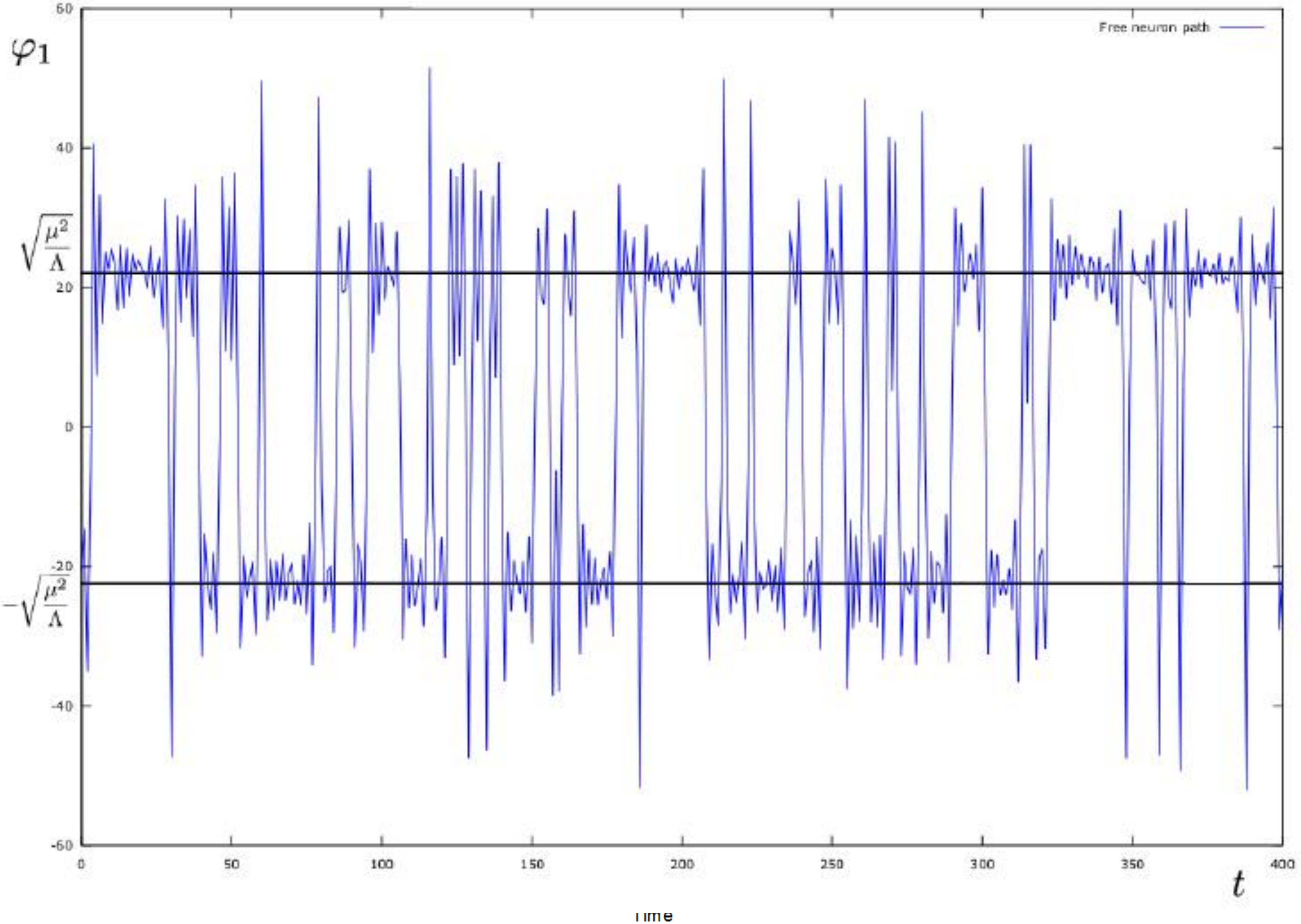

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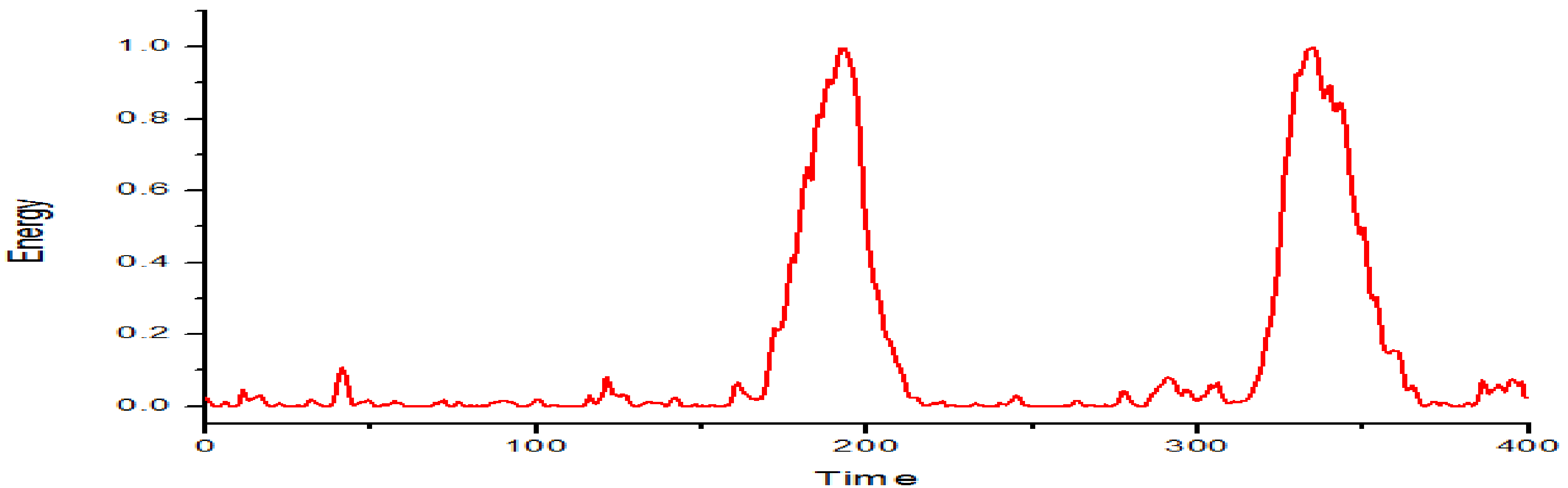
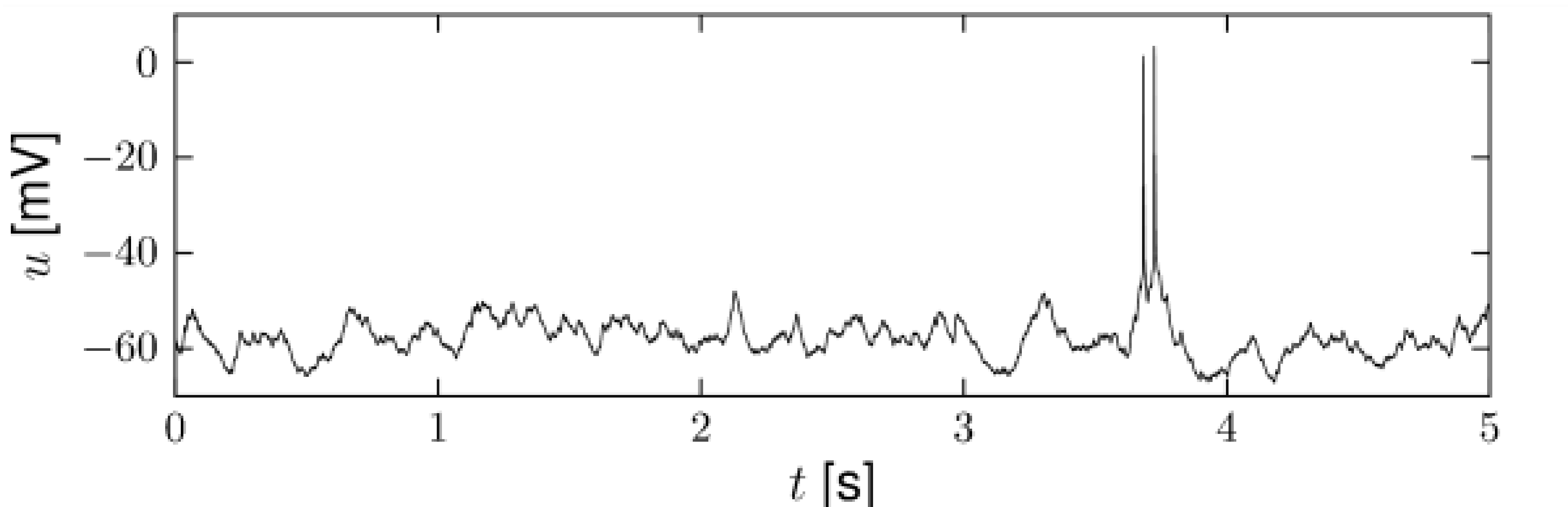
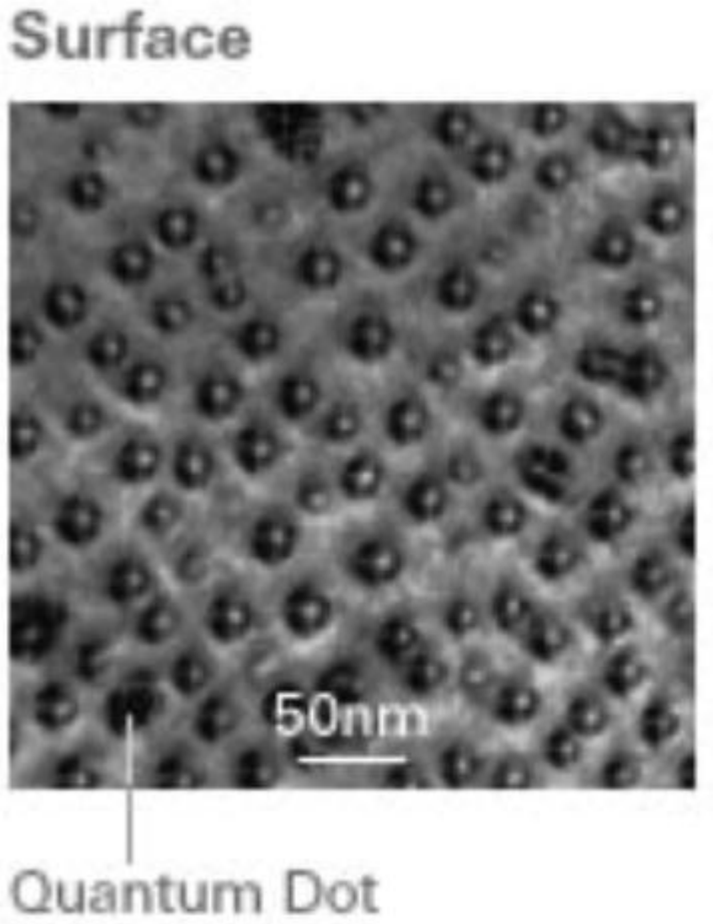




•Nano-technological realizations

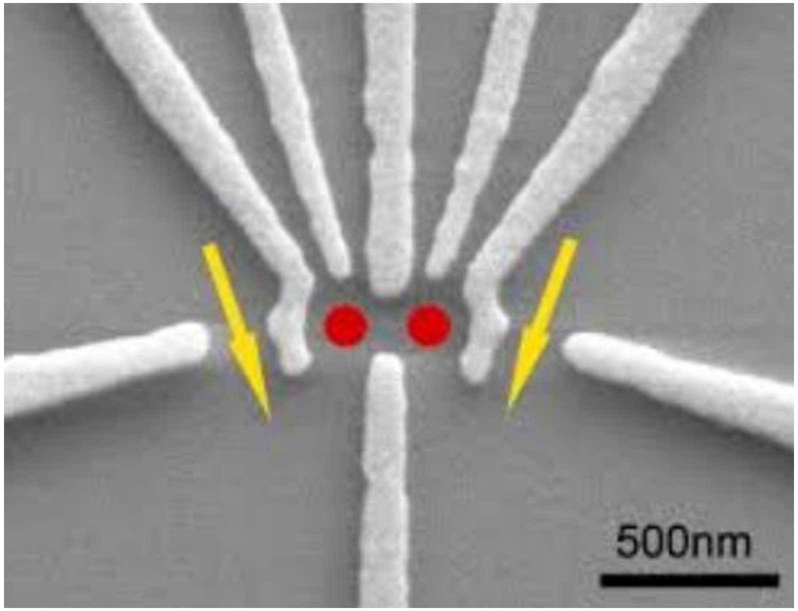
•We need in Nano-technological platform for realization of QNN.

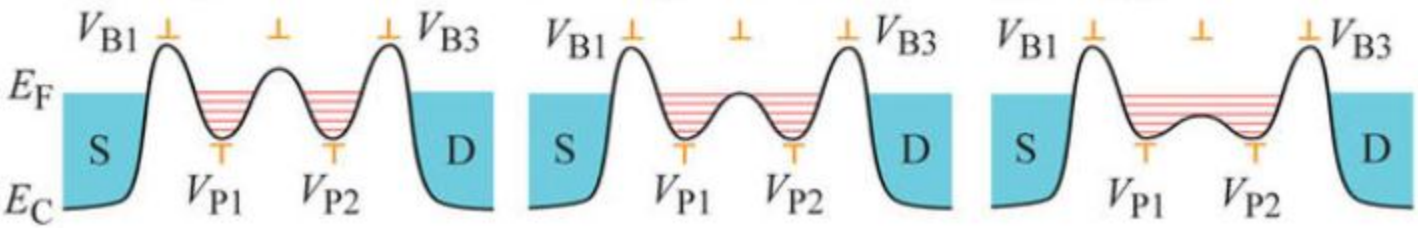
•One possible way: **quantum double dots**.



•Nano-technological realizations

•**Quantum double dots**.

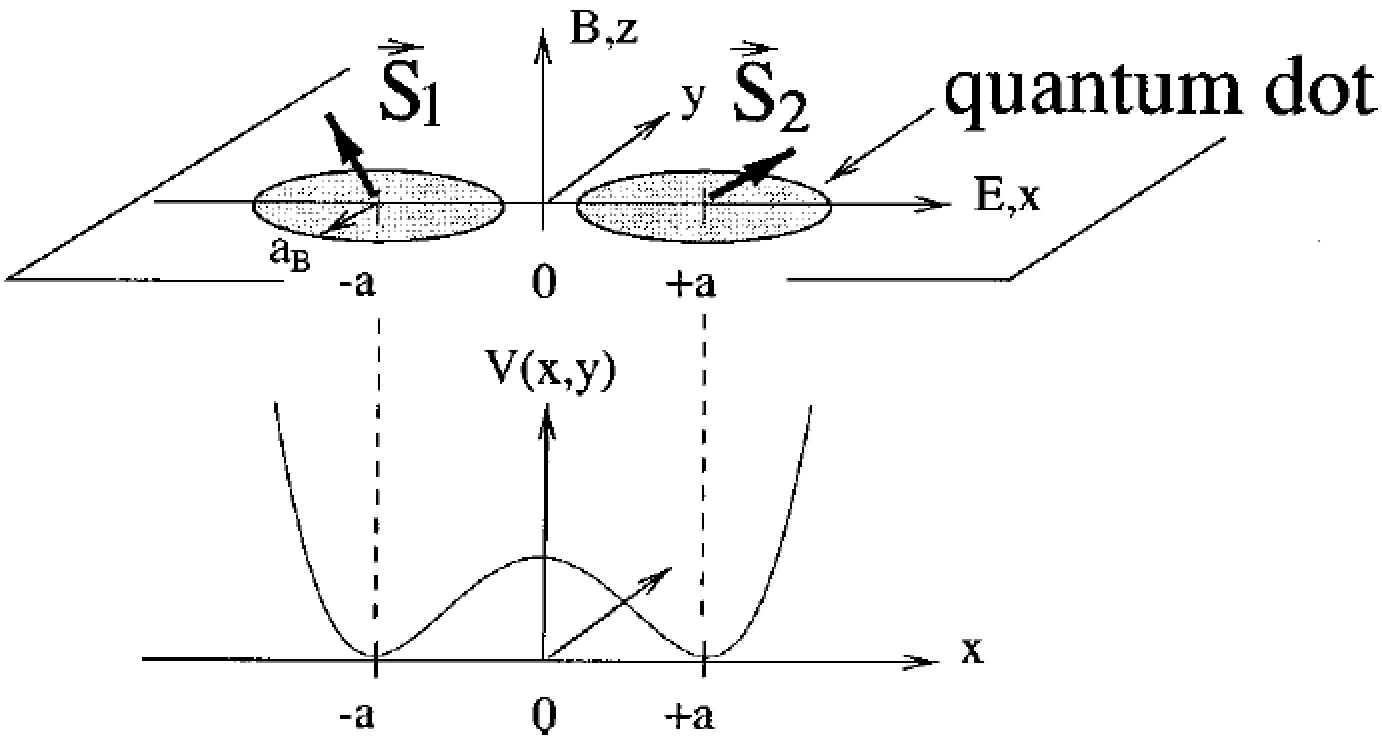




•*Scientific Reports* **1**, Article number: 110 (2011)

•Nano-technological realizations

•**Quantum double dots**.



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Coupled quantum dots as quantum gates

Guido Burkard* and Daniel Loss†
Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland

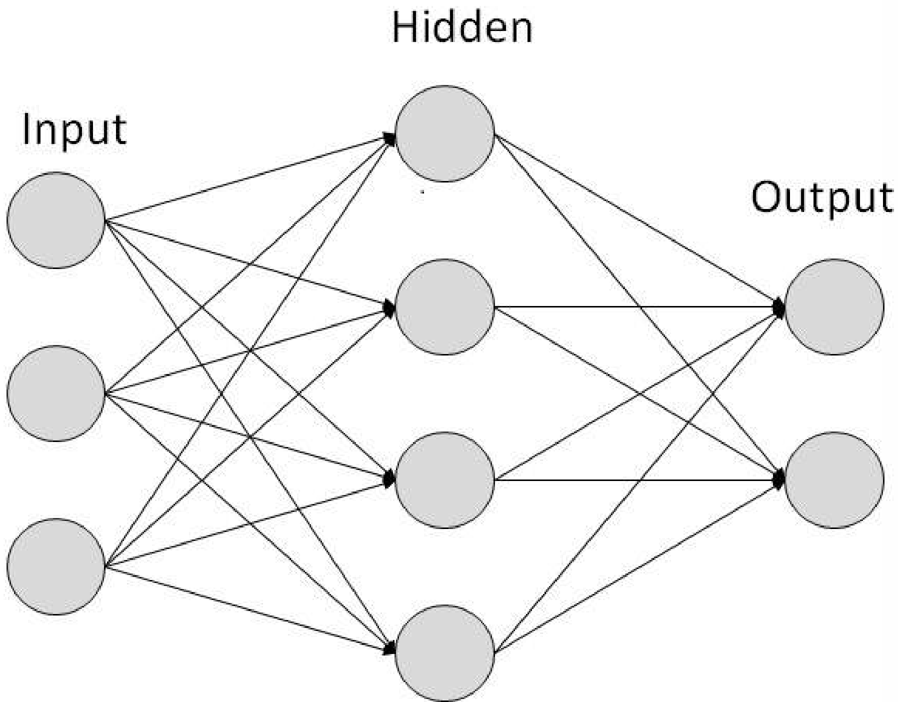
David P. DiVincenzo‡
IBM Research Division, Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598

(Received 3 August 1998)

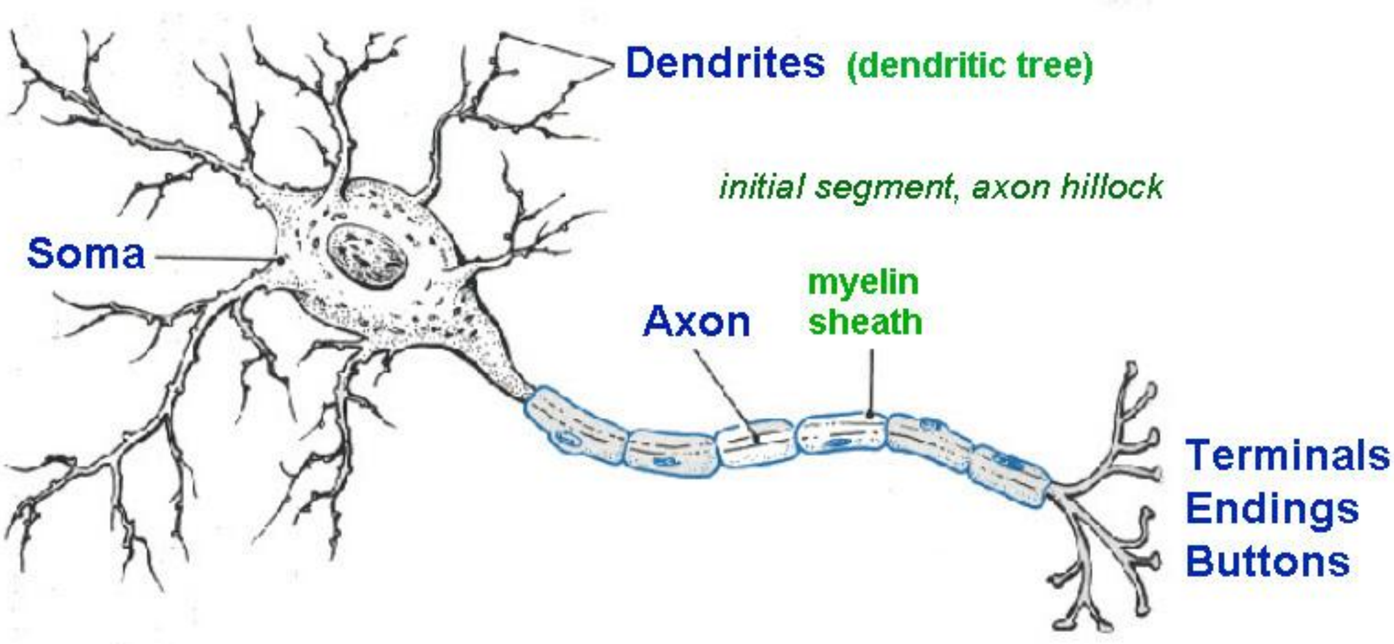
•Quantum neural network as quantum many body system

$$Z = \int \prod_i \mathcal{D}q_i\,(\tau) \exp(-\frac{S(q_i(\tau))}{\hbar}), q_i(0) = q_i(T),$$

$$S = \int_0^T d\tau \left[\sum_i \left(\frac{1}{2}\hat{p}_i^2 + V_0(\hat{q}_i) \right) + \sum_{i>j} V_{int}(\hat{q}_i, \hat{q}_j) \right].$$



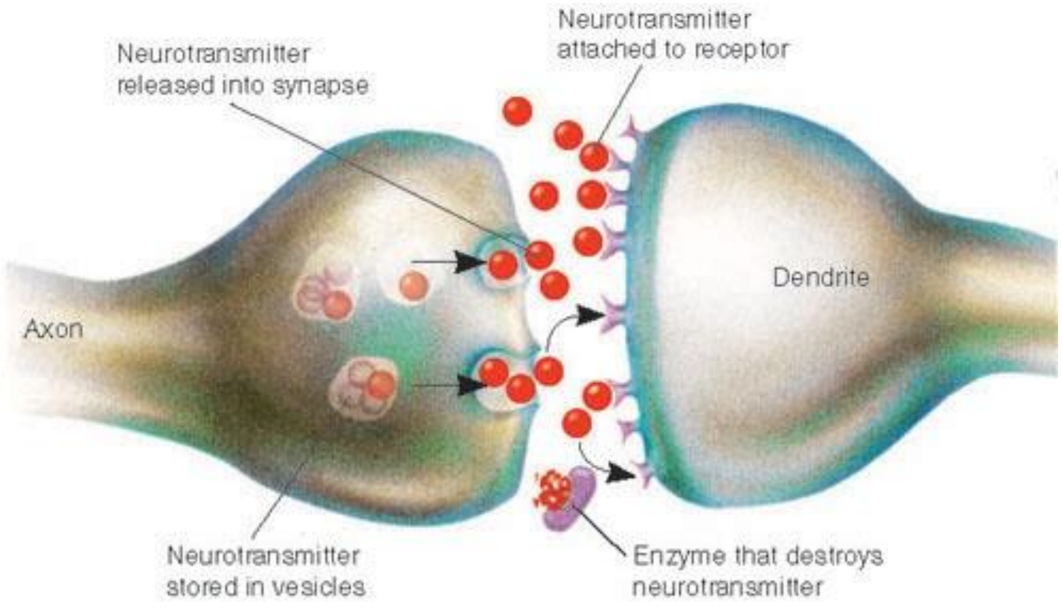
•Axons in neural net



- **"Axon" is output information line from neuron.**
- **So neural net is very non-local system.**

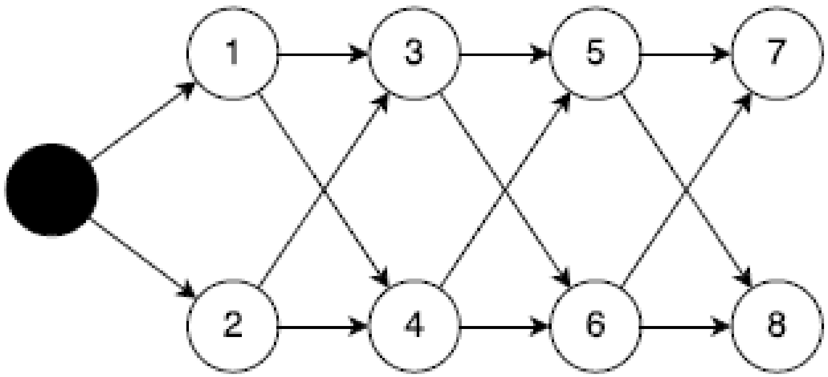
•Role of Synepse

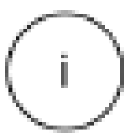
•Role of Synepse is the contact coefficient, the measure of neuron connection.



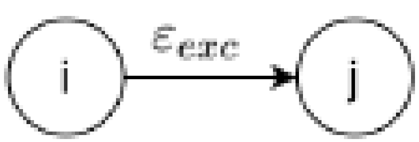
•Excitation connection

$$S = \int_0^T d\tau \left[\sum_i \left(\frac{1}{2}\hat{p}_i^2 + V_0(\hat{q}_i) \right) + \sum_{i>j} V_{int}(\hat{q}_i, \hat{q}_j) \right].$$



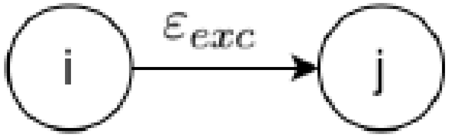
1. 

$- \mathcal{L}_0 = \frac{1}{2}\dot{\varphi}_i^2 + \frac{\Lambda}{4}\left(\varphi_i^2 - \frac{\mu^2}{\Lambda}\right)^2$

2. 

$- \mathcal{L}_{int} = \varepsilon_{exc}\varphi_j^2\left(\varphi_i^2 - \frac{\mu^2}{\Lambda}\right)^2$

•Excitation connection



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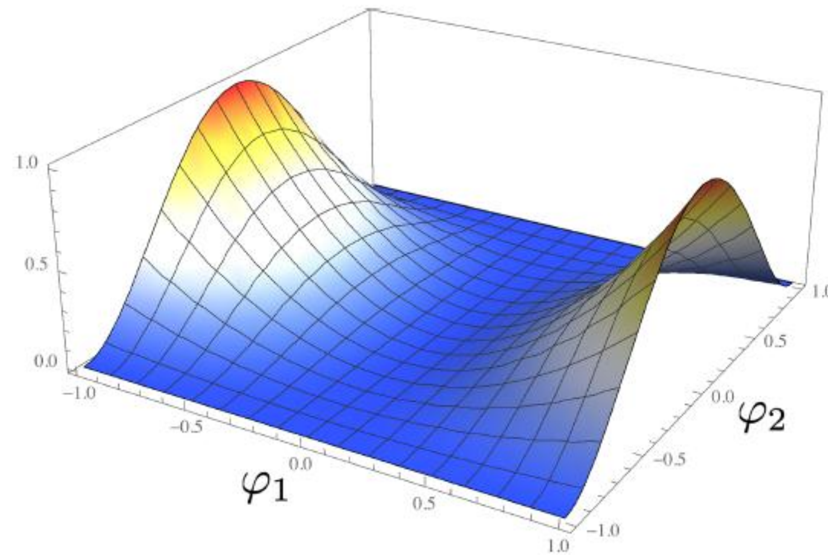
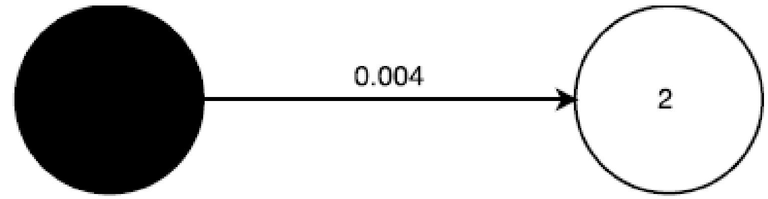


•Excitation connection: simple test



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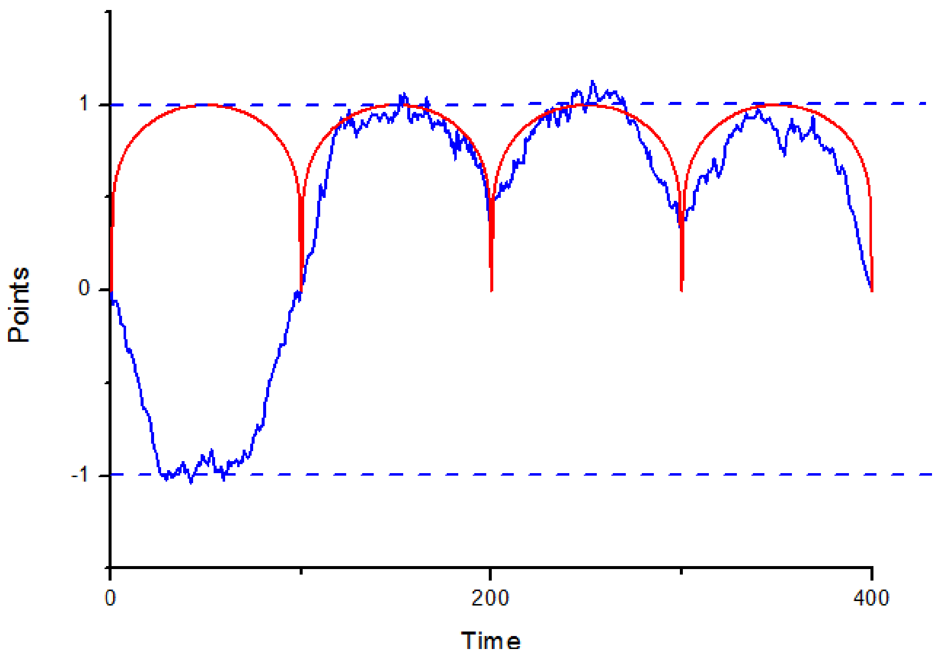

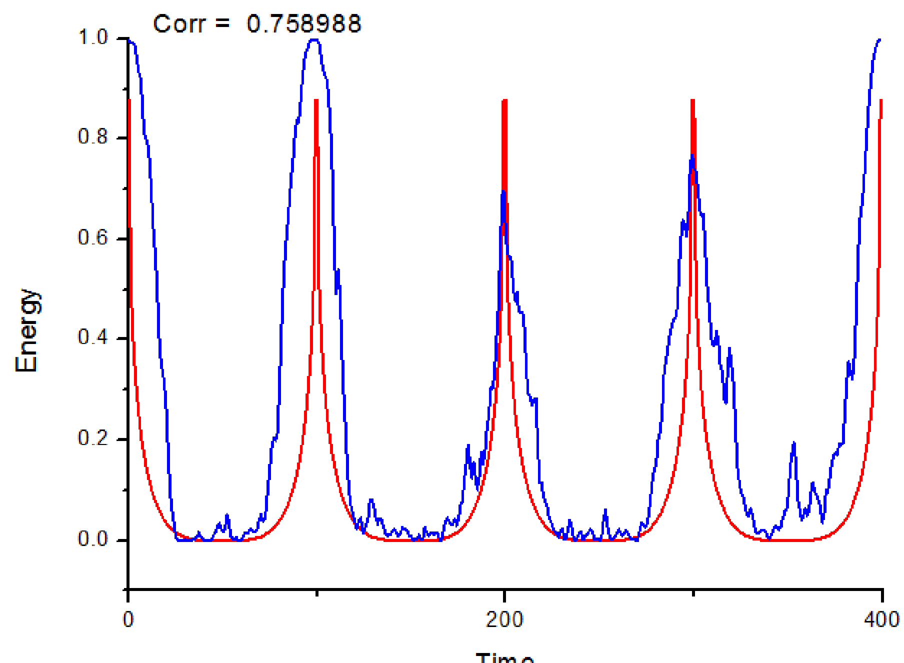

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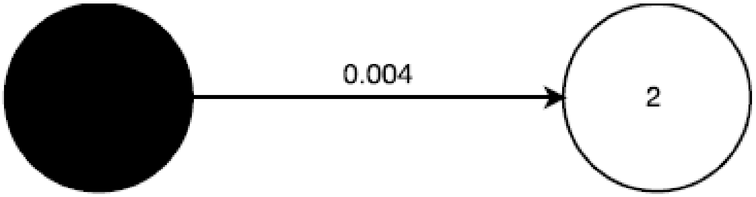
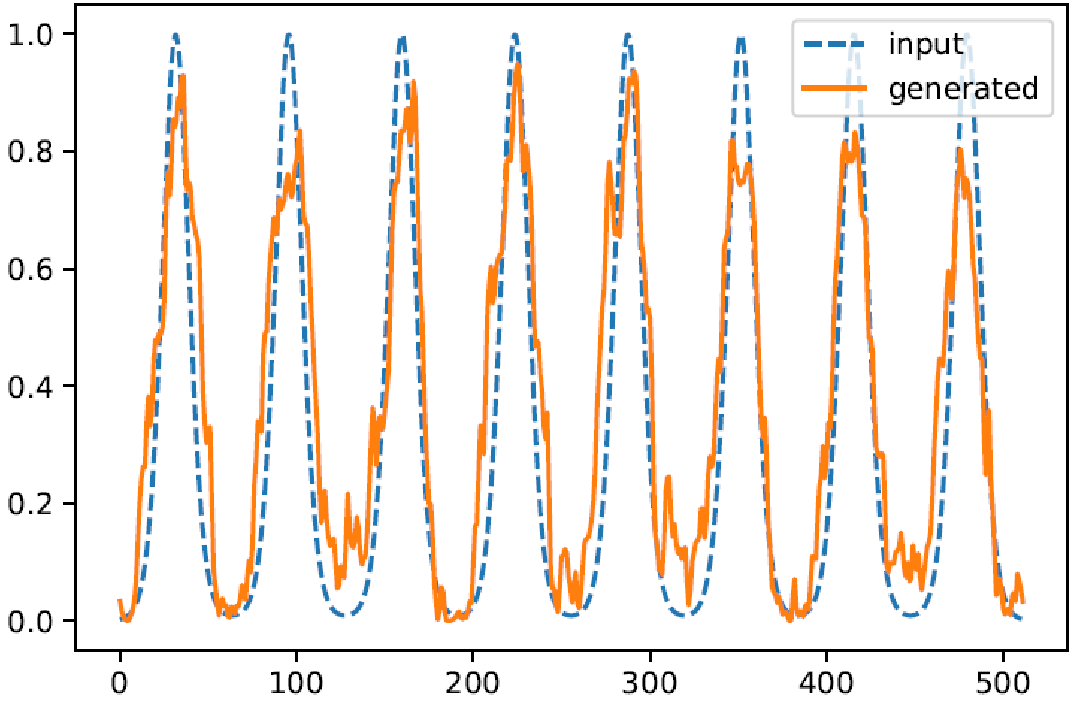
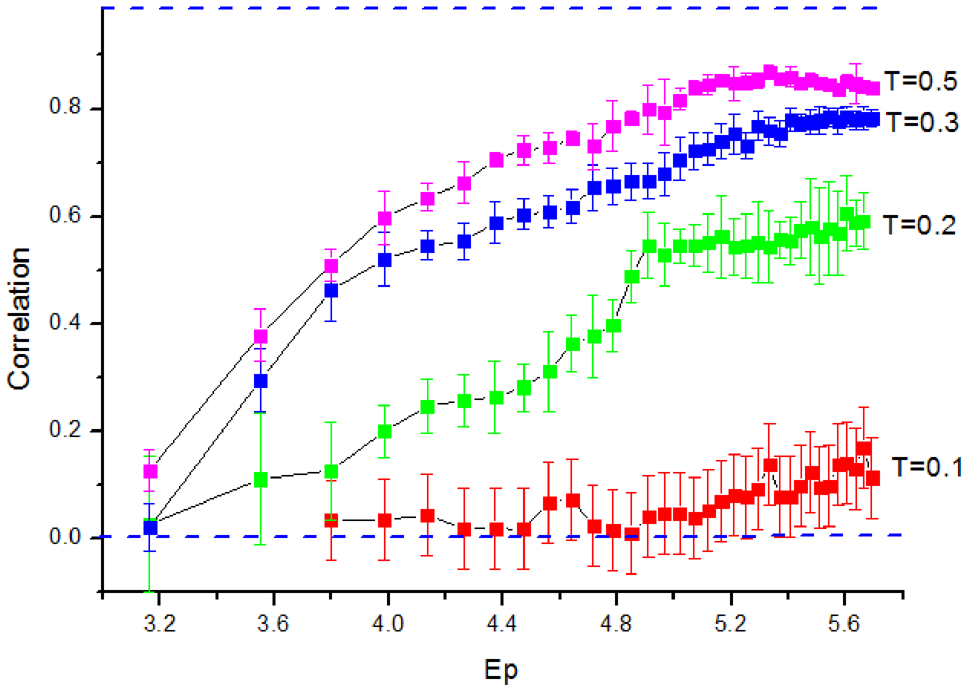


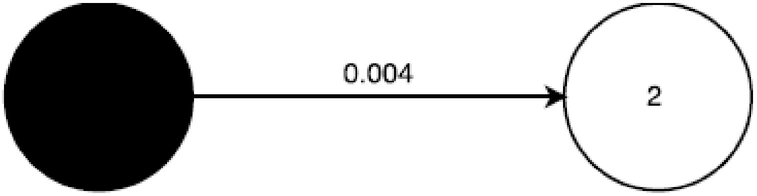
•Activity of Q-neuron

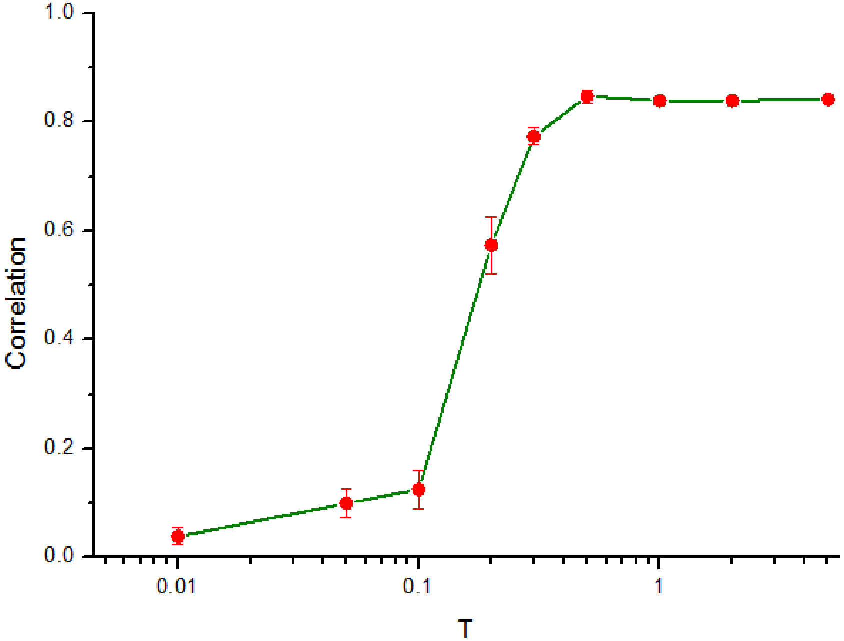
$$\left(\varphi_2^2 - 1\right)^2$$



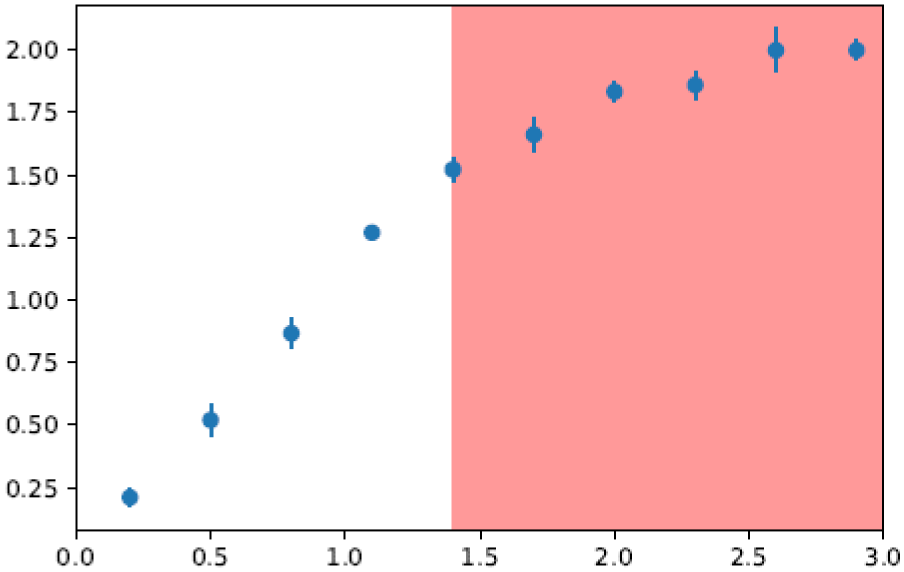
•Quantum neuron

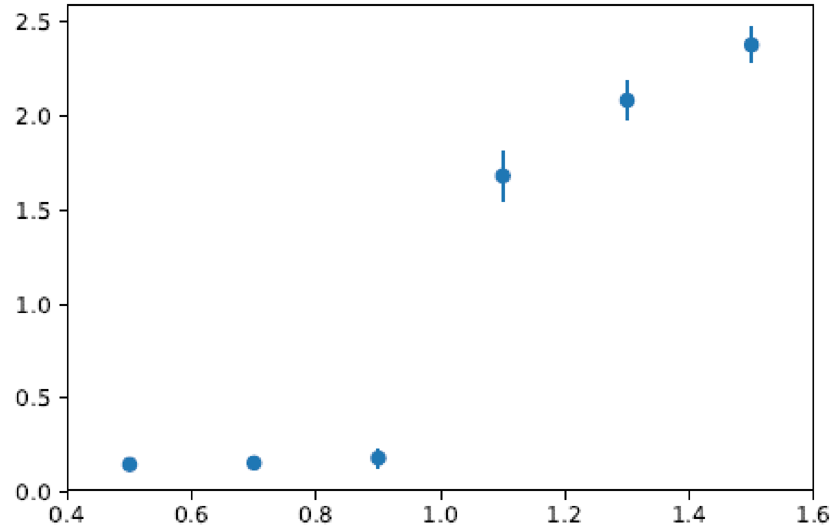


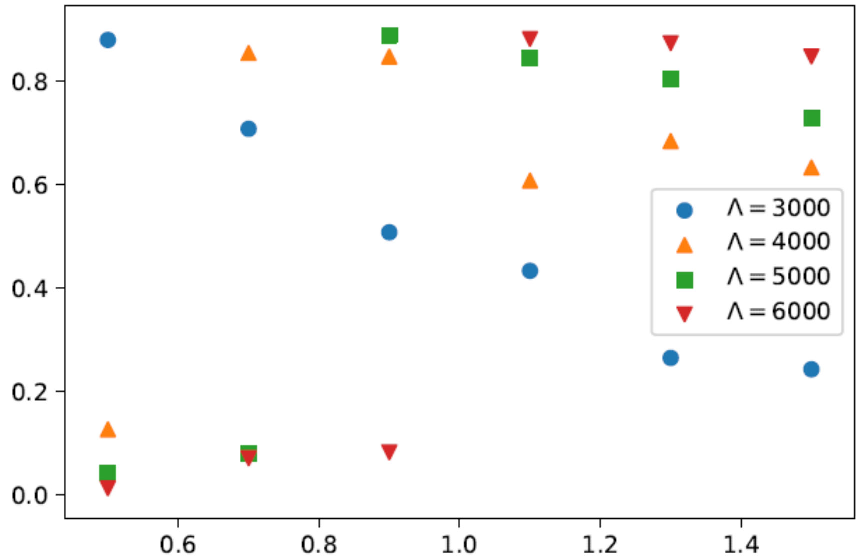




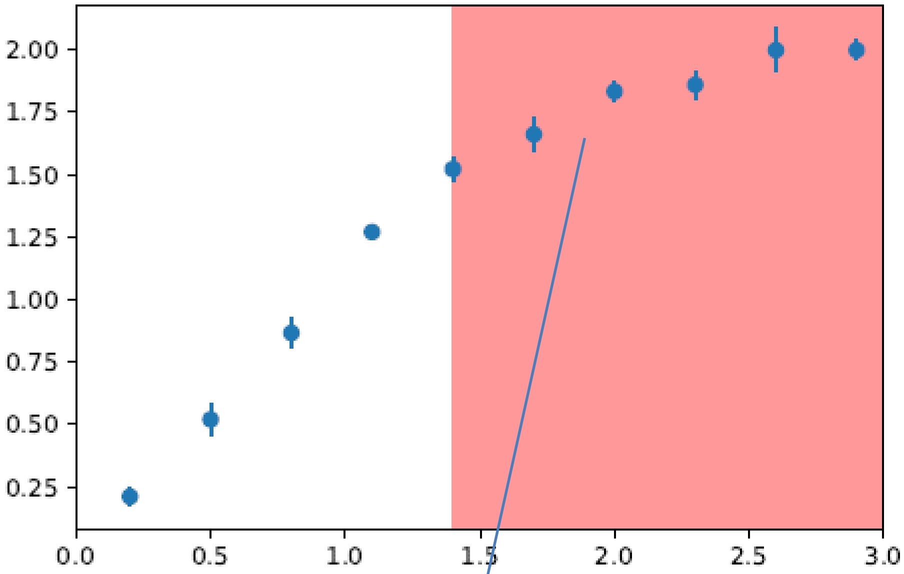
•Excitation connection: 3 Q-neurons transport

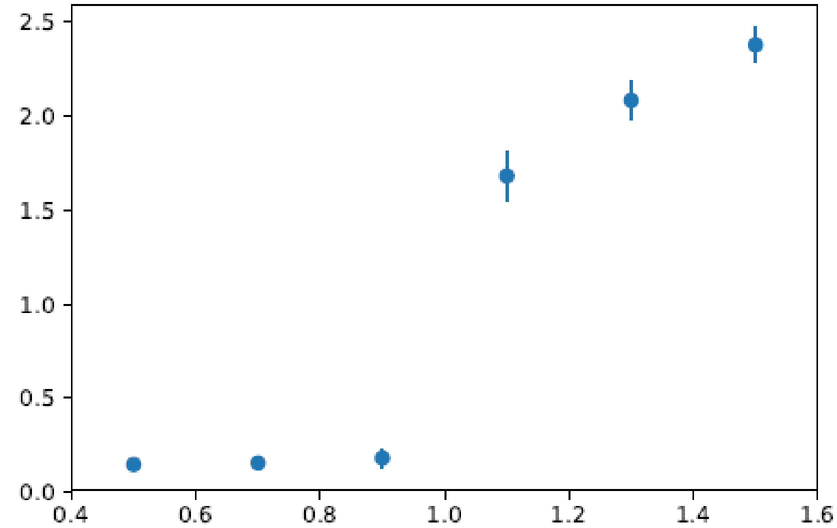


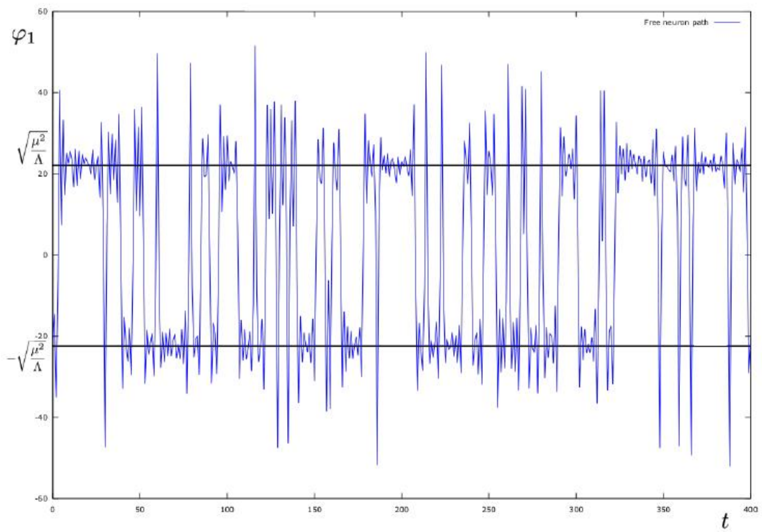




•Phase transition and Epileptic seizure

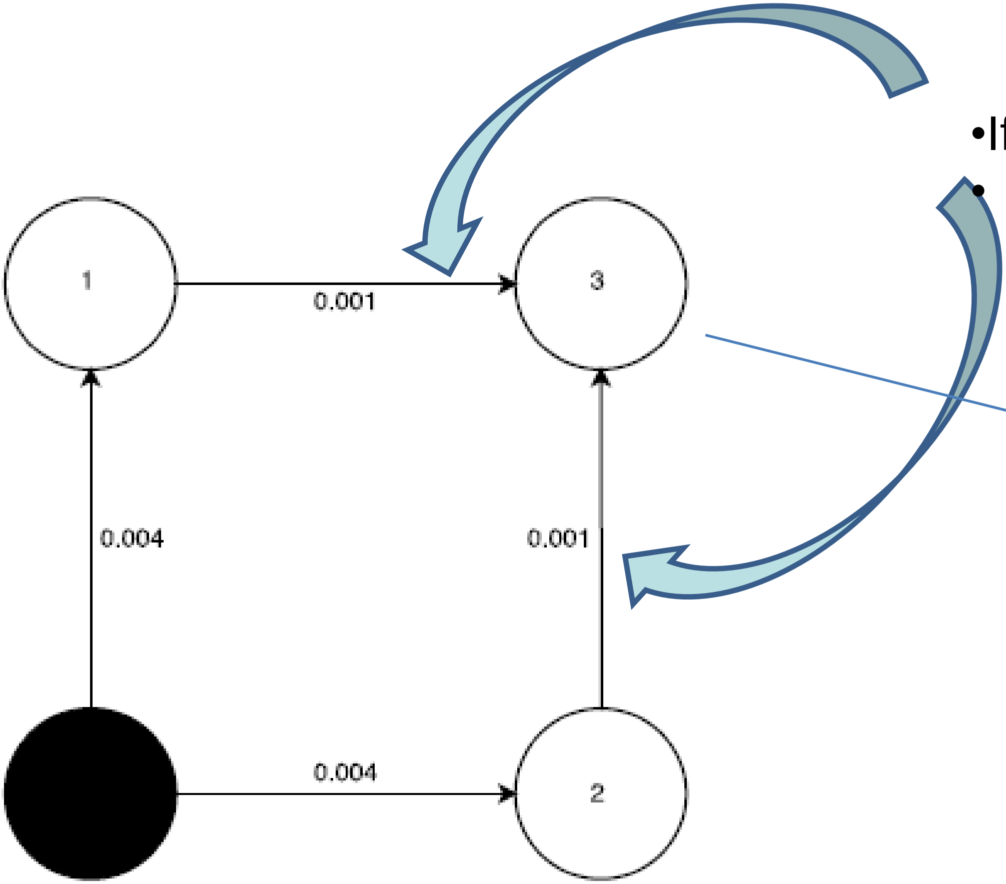






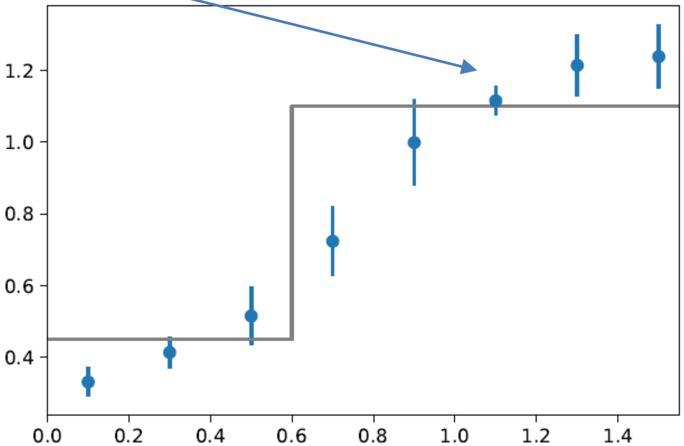
•Quantum neural network logical elements

•Logical AND



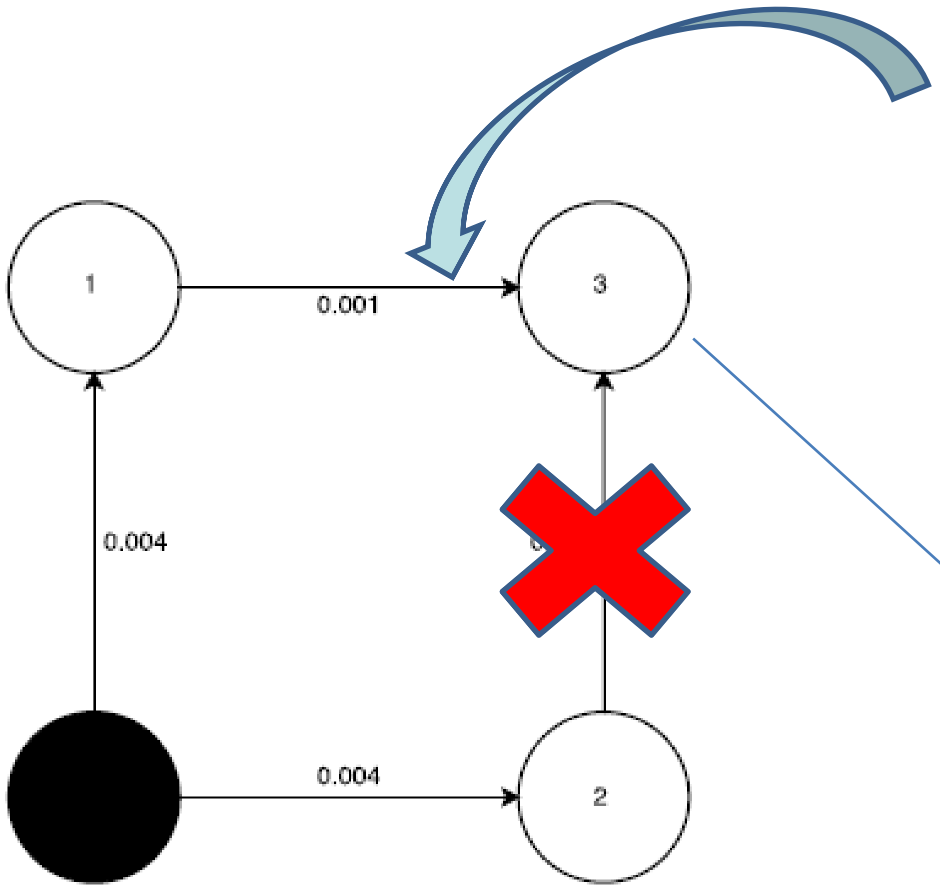
•If both Paths are active,

• Correlation more 70 %



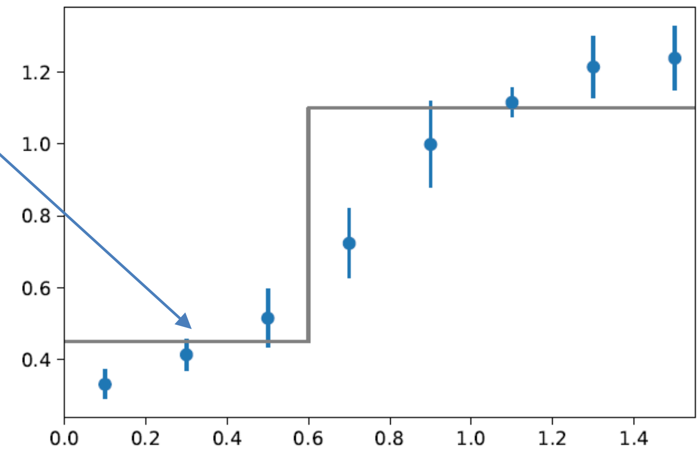
•Quantum neural network logical elements

•**Logical AND**

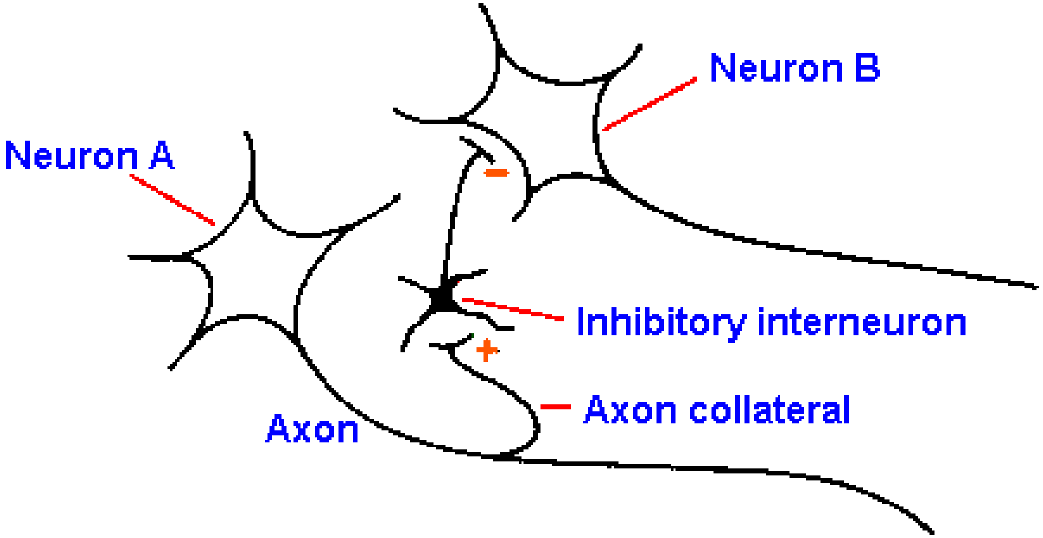


•If only one Path is active,

• Correlation less 20 %

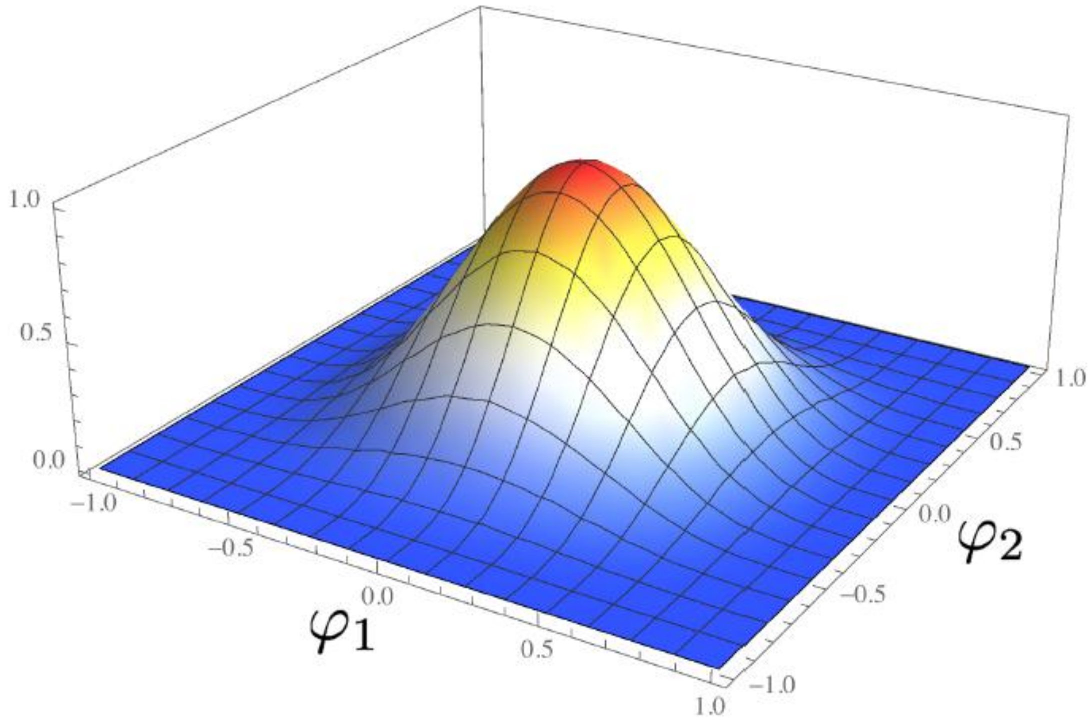


•Inhibiting potential

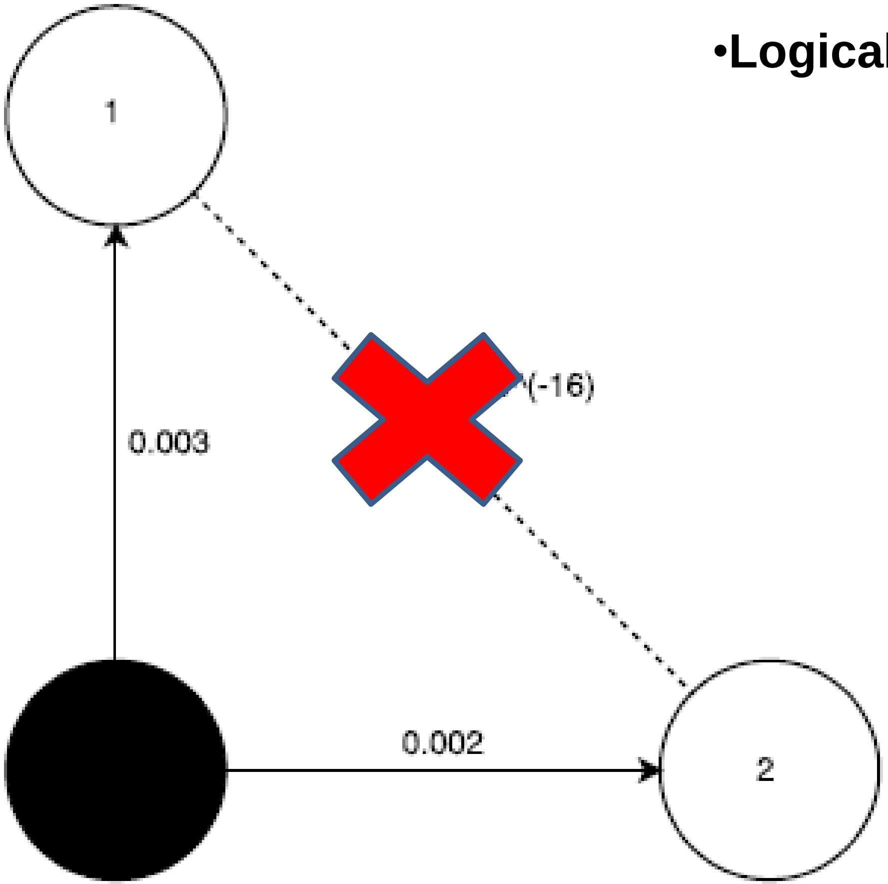


•Inhibiting potential

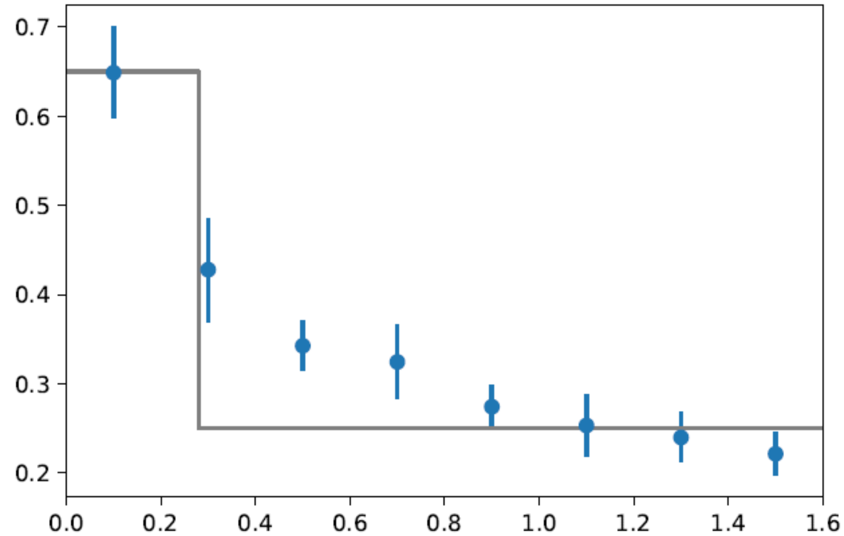
$$-\mathcal{L}_{int} = \varepsilon_{inh}\left(\varphi_i^2 - \frac{\mu^2}{\Lambda}\right)^4\left(\varphi_j^2 - \frac{\mu^2}{\Lambda}\right)^4$$



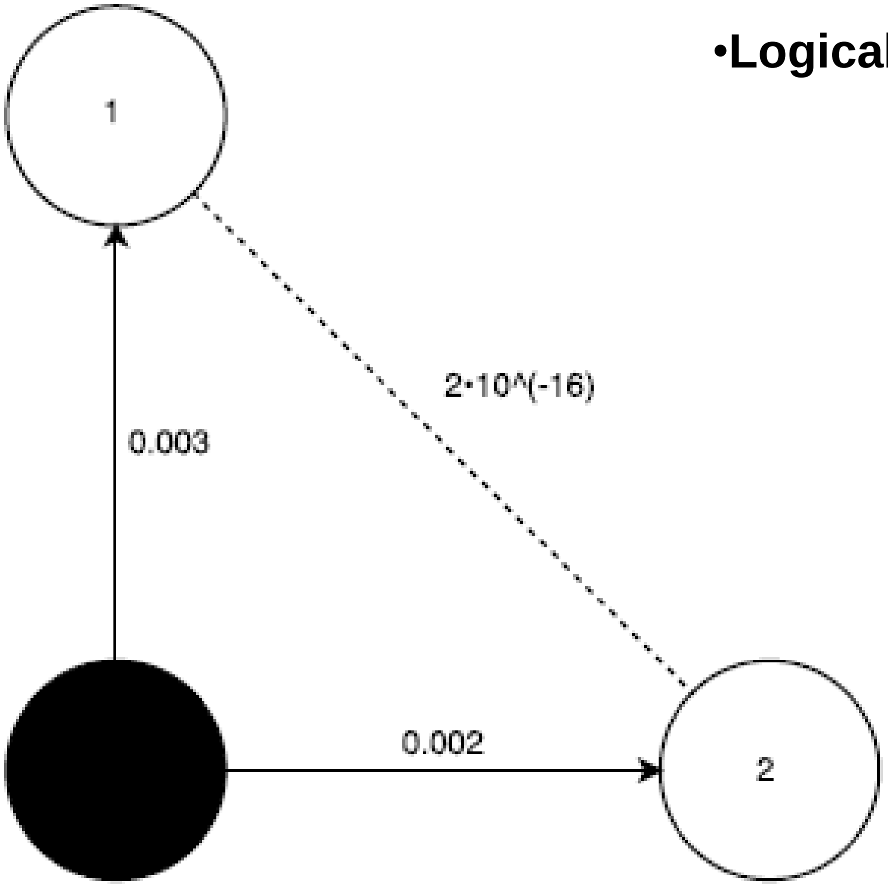
•Quantum neural network logical elements



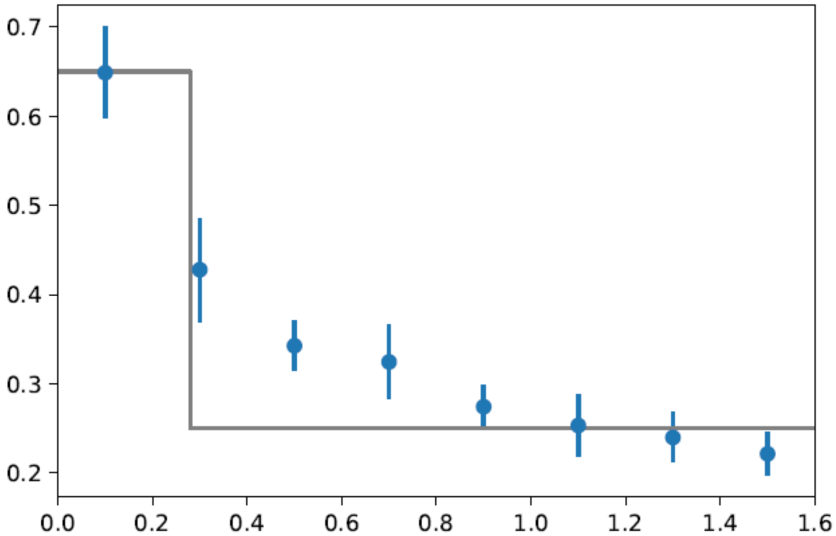
•Logical NOT



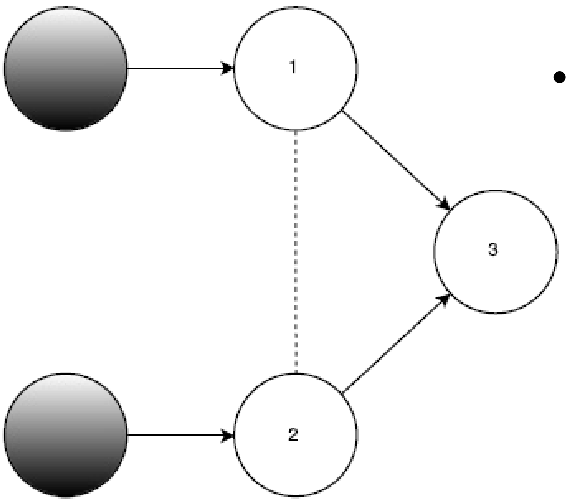
•Quantum neural network logical elements



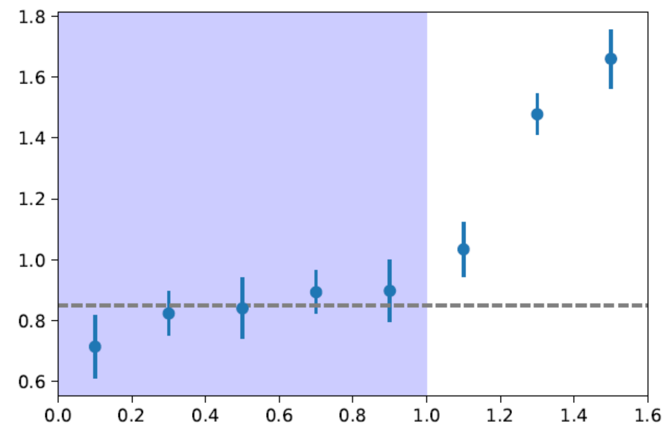
•Logical **NOT**



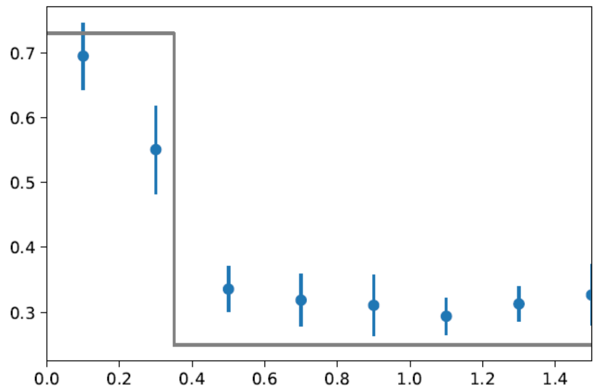
•Quantum neural network logical elements

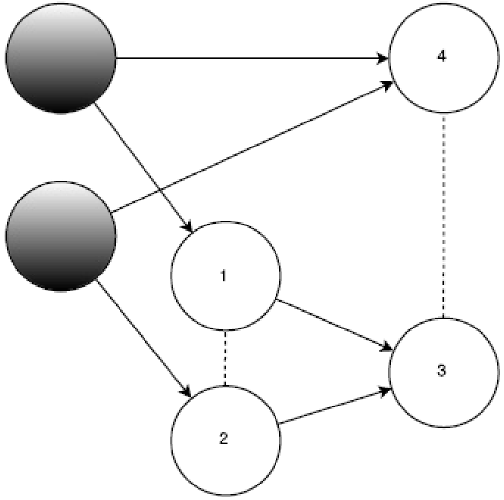


•Logical **OR**

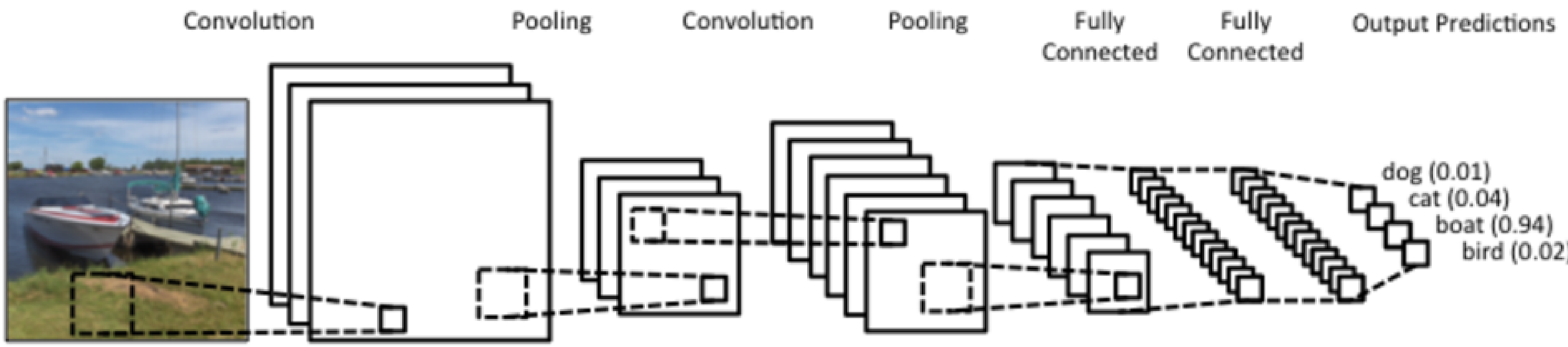


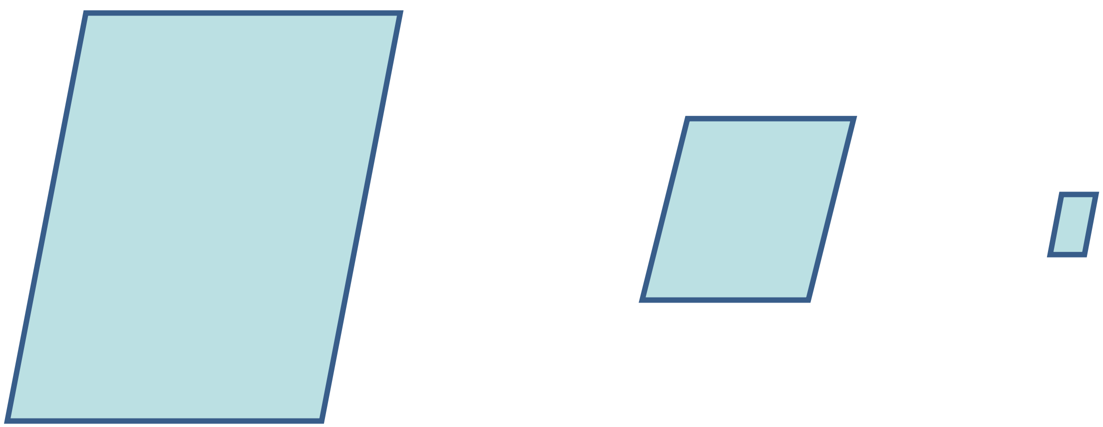
•Logical **exclusive OR (XOR)**



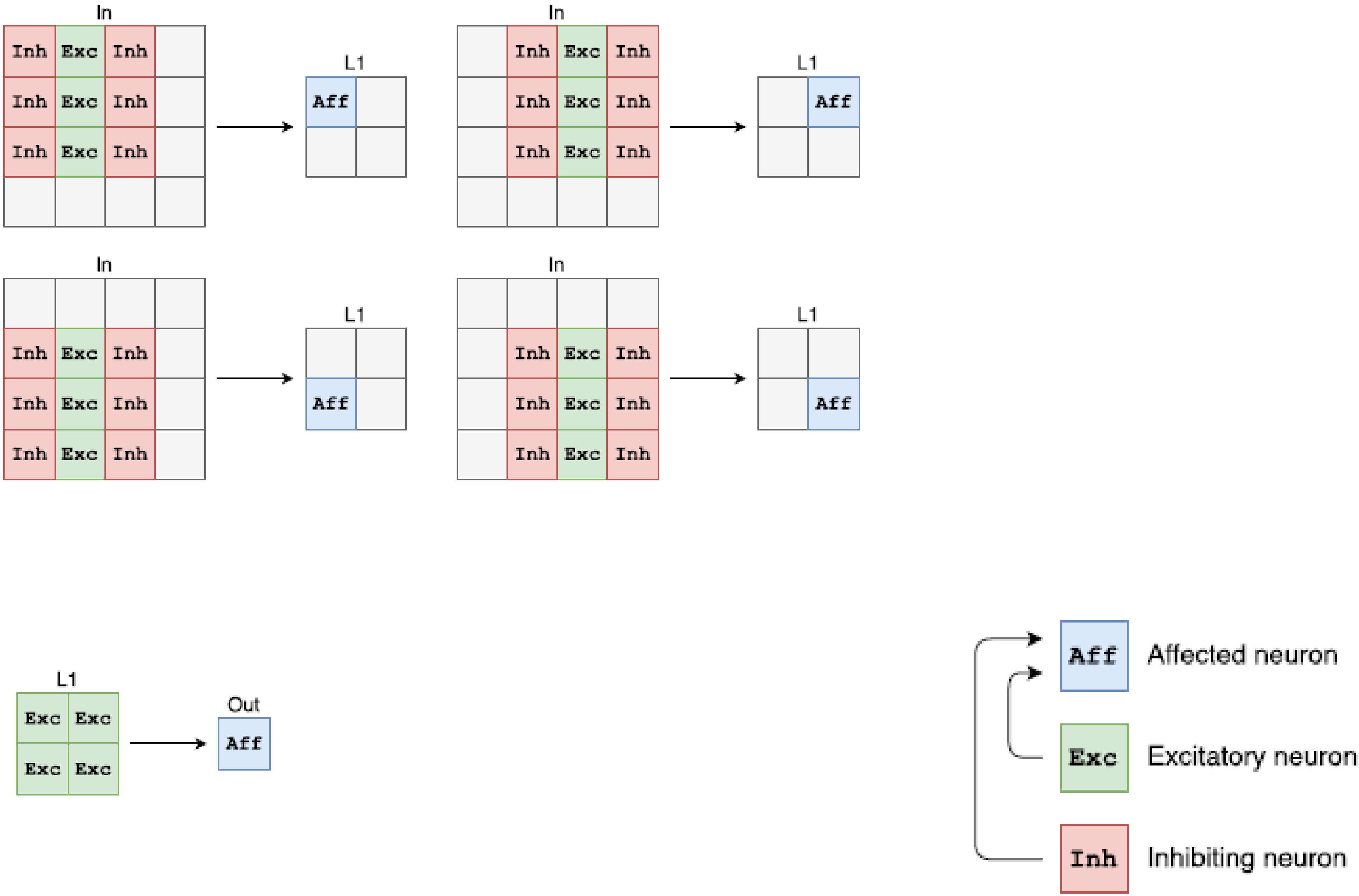


•Convolution neural network

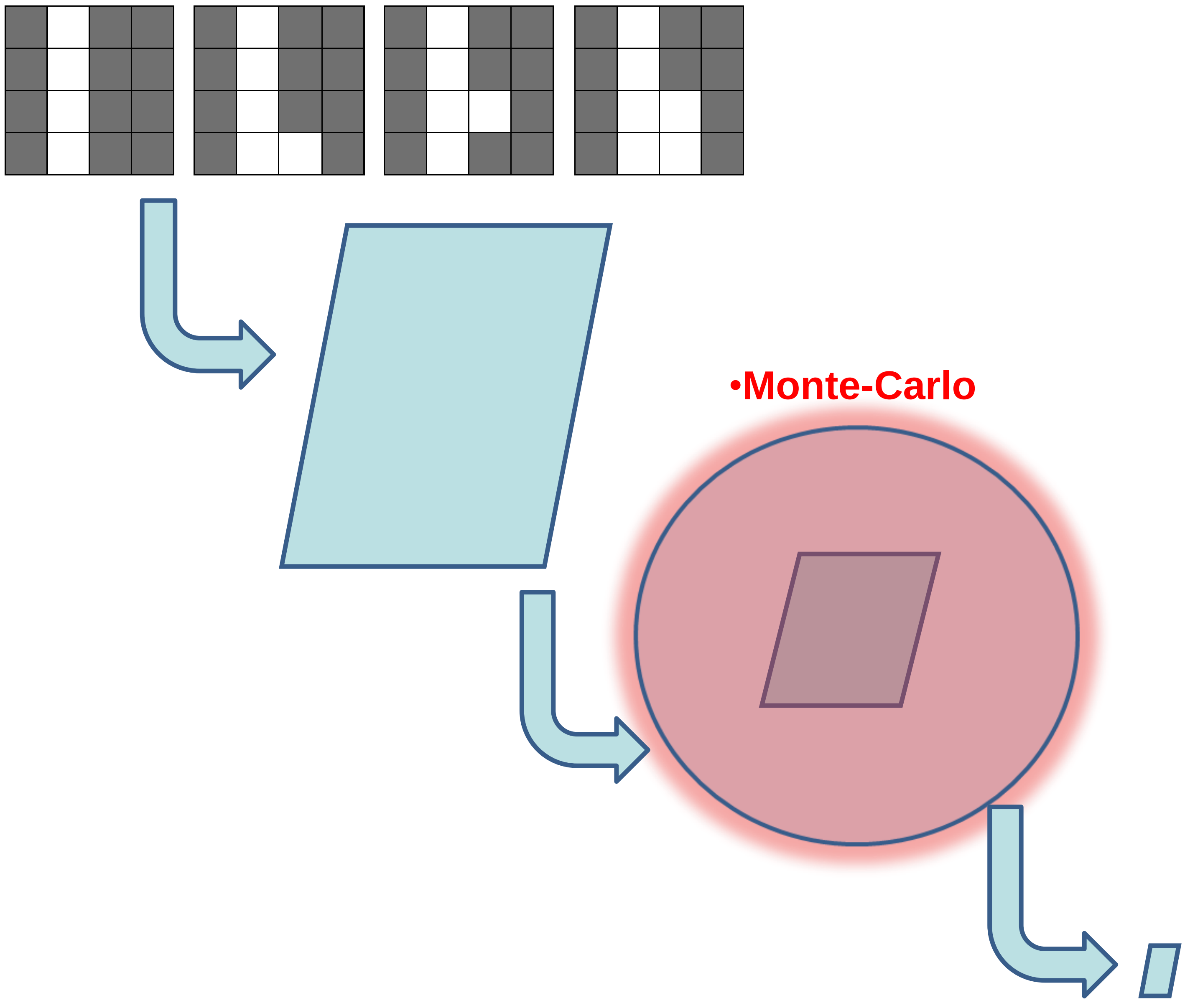




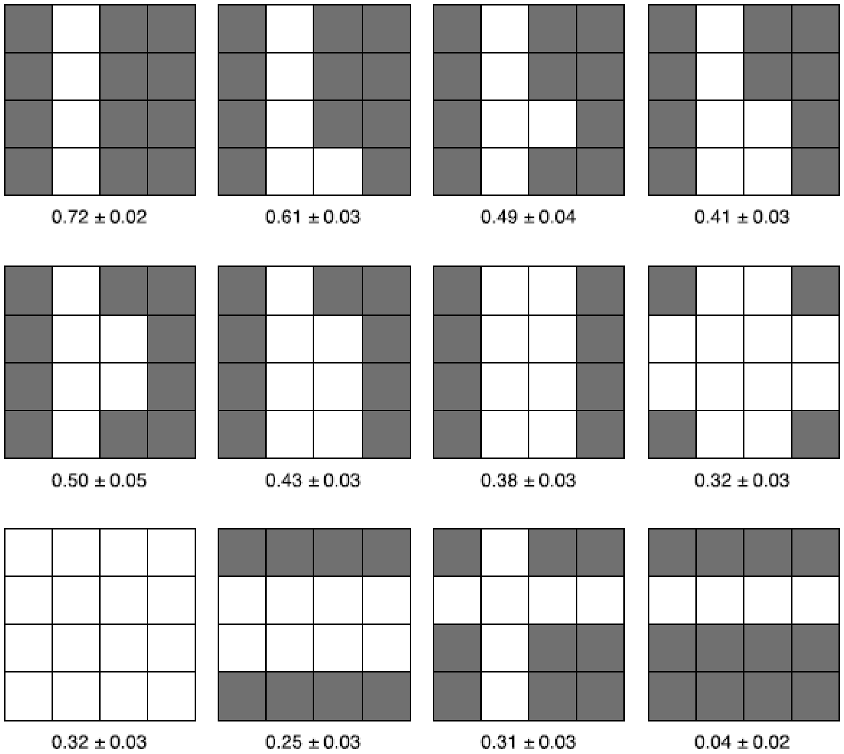
•Convolution neural network

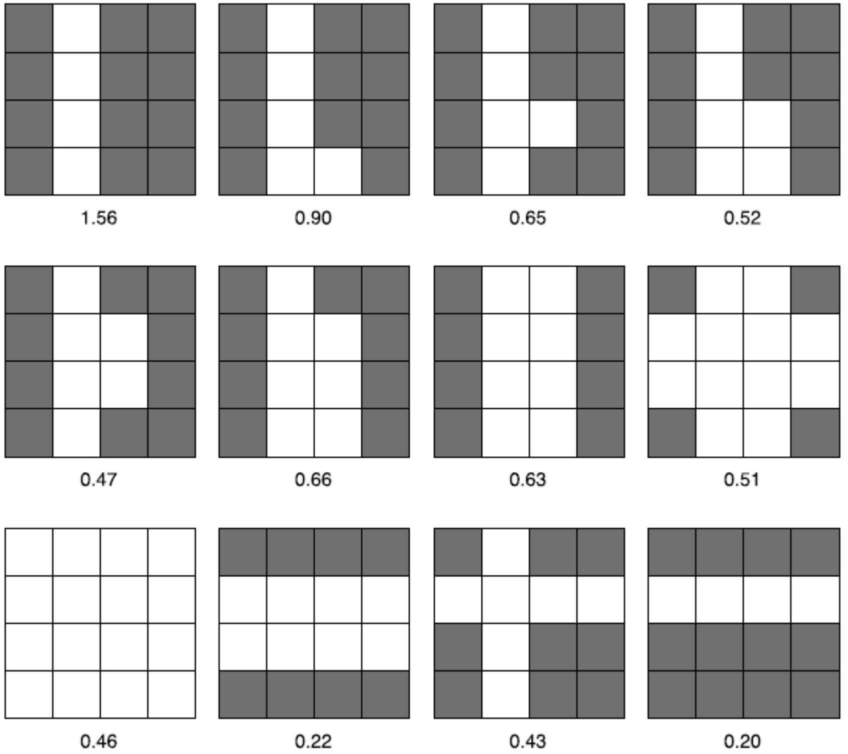


•Convolution neural network



•Convolution neural network

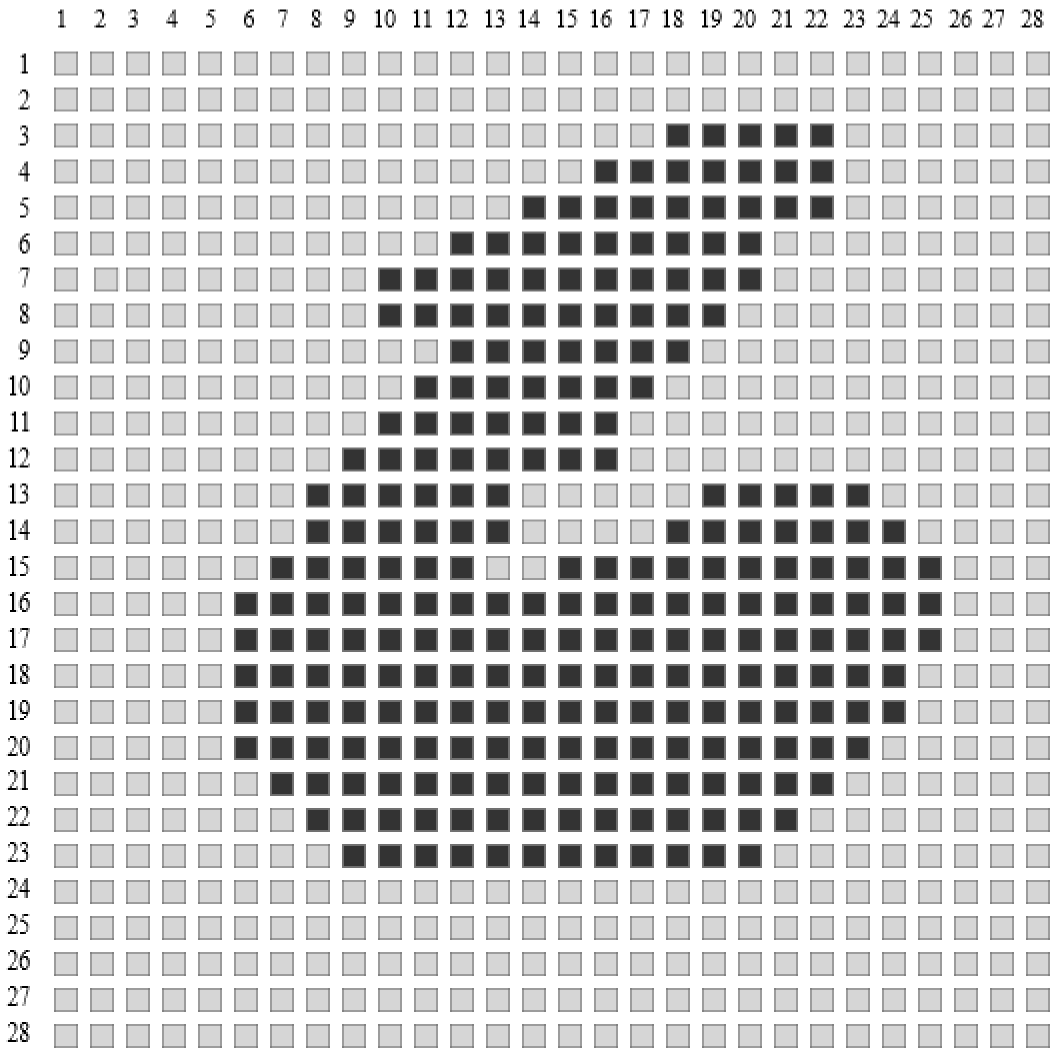




•Digit recognition

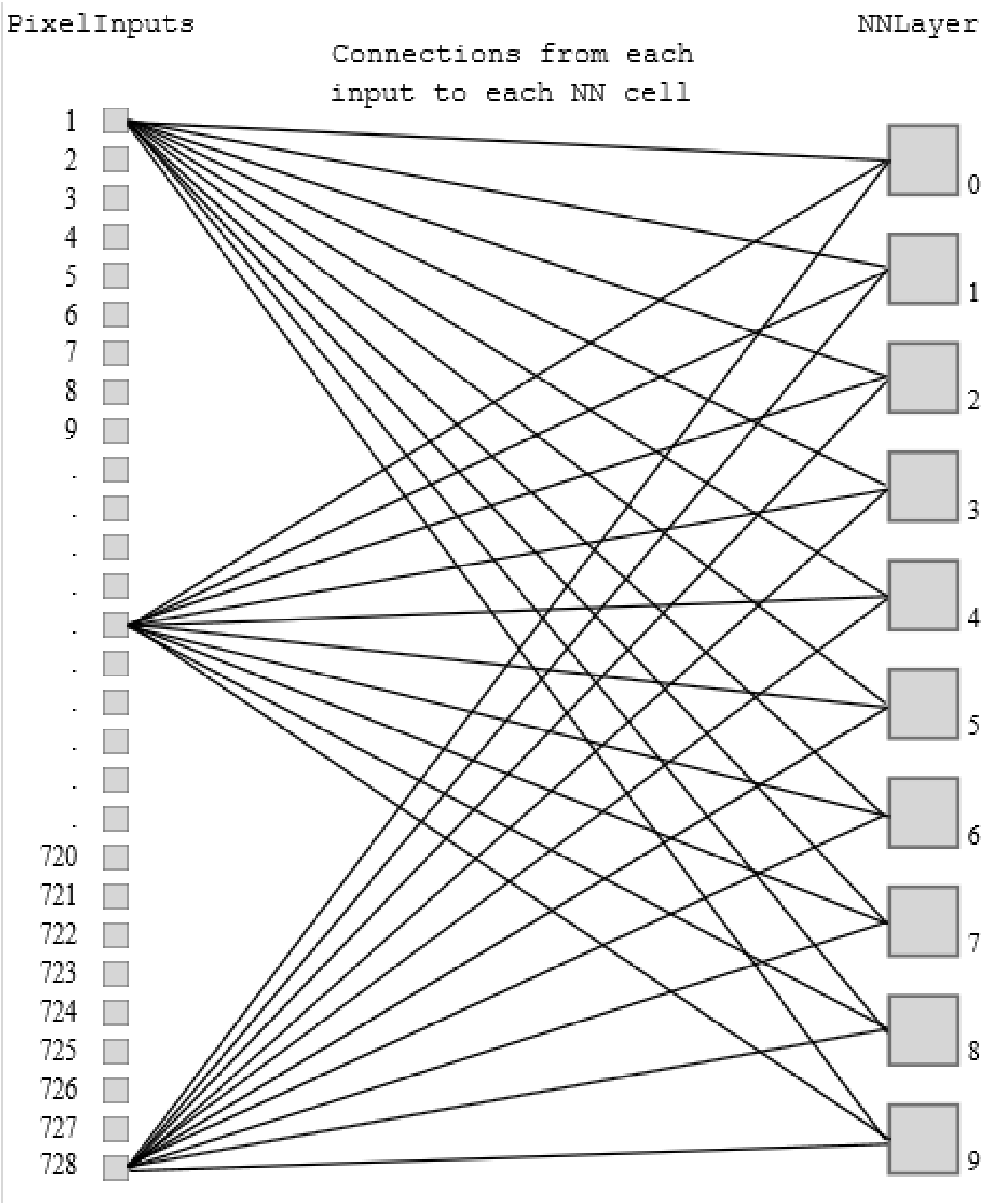
•MNIST database

•Digit recognition



•**MNIST database**: **MNIST image has a size of 28*28 = 784 pixels**

•Digit recognition



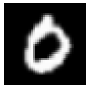
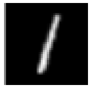
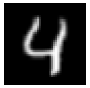
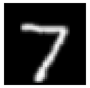
•Digit recognition

$$\mathcal{L}_0 = \sum_{i=0}^{784}\left[\frac{1}{2}\dot{\psi}_i^2 + \frac{\Lambda}{4}\left(\psi_i^2 - \frac{\mu^2}{\Lambda}\right)^2\right] + \sum_{j=0}^{10}\left[\frac{1}{2}\dot{\varphi}_j^2 + \frac{\Lambda}{4}\left(\varphi_j^2 - \frac{\mu^2}{\Lambda}\right)^2\right]$$

$$\mathcal{L} = \mathcal{L}_0 + \sum_{i=0}^{784}\sum_{j=0}^{10} k\left(\varepsilon_{ij} - b\right) A_i \varphi_j^2 \left(\psi_i^2 - \frac{\mu^2}{\Lambda}\right)^2 + \\ + 10^{-17}\sum_{k>j}^{10}\sum_{j=0}^{10}\left(\varphi_j^2 - \frac{\mu^2}{\Lambda}\right)^4\left(\varphi_k^2 - \frac{\mu^2}{\Lambda}\right)^4,$$

$$Z = \int \prod_i \mathcal{D}q_i\left(\tau\right)\exp\left(-\frac{S(q_i(\tau))}{\hbar}\right), q_i(0) = q_i(T),$$

•Digit recognition

				
P(0) = 0.338608	P(1) = 0.655741	P(2) = 0.451795	P(4) = 0.362327	P(7) = 0.605863
P(6) = 0.13097	P(8) = 0.0840482	P(6) = 0.207845	P(8) = 0.16814	P(9) = 0.153816
P(7) = 0.104982	P(3) = 0.0834241	P(3) = 0.12158	P(2) = 0.14839	P(8) = 0.0527513
P(2) = 0.0962352	P(2) = 0.0605042	P(5) = 0.0695778	P(1) = 0.104967	P(3) = 0.0501808
P(5) = 0.0873339	P(7) = 0.0424982	P(0) = 0.0440336	P(9) = 0.0852715	P(1) = 0.0482902
P(4) = 0.0781002	P(0) = 0.037828	P(8) = 0.0399262	P(3) = 0.0759215	P(0) = 0.0334645
P(9) = 0.0714371	P(4) = 0.0180404	P(9) = 0.0267228	P(5) = 0.0338338	P(5) = 0.0295938
P(3) = 0.0662971	P(9) = 0.0130295	P(7) = 0.0263385	P(6) = 0.0196095	P(2) = 0.0167923
P(8) = 0.0260371	P(6) = 0.00488656	P(4) = 0.012181	P(0) = 0.00153889	P(4) = 0.00924887
P(1) = 0	P(5) = 0	P(1) = 0	P(7) = 0	P(6) = 0

Euclidian Quantum Field Theory and spin models

Let us consider scalar field $\varphi_\alpha(x,t)$, $\alpha = 1,2$

Action of φ^4-model in 1+1 dim:

$$S[\varphi_\alpha] = \int dx\,dt \left[\frac{1}{2}\partial_\mu \varphi_\alpha \partial_\mu \varphi_\alpha + \frac{\lambda}{4}\left((\varphi_\alpha\varphi_\alpha)^2 - \frac{\mu^2}{\lambda}\right)^2 \right]$$

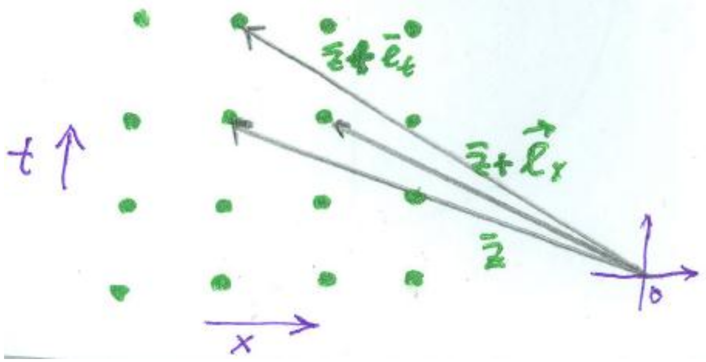
$$Z = tr_E\, e^{-\beta H} = \int\limits_{\varphi(0)=\varphi(\beta)} \mathcal{D}\varphi\, e^{-S(\varphi)}$$

↑ O(2) symmetry!

(x,t) lattice:

$x: a_x n_x$ | $t: a_t n_t$ | $\varphi_{n_x,n_t} = \varphi_{\bar z} : \bar z = (a_x n_x, a_t n_t)$

$$\bar\mu = \begin{cases} \mu = 1 : a_x \bar n_x \\ \mu = 2 : a_y \bar n_y \end{cases} \Big| \rightarrow \partial_\mu \varphi_\alpha = \frac{\varphi_{\alpha,\bar z+\bar\mu} - \varphi_{\alpha,z}}{a_\mu}$$



Let $a_x = a_t = 1$

So: $\mathcal{D}\varphi_\alpha \Rightarrow \prod\limits_{\bar z} d\varphi_{\alpha,\bar z}$

$$\partial_\mu \varphi_\alpha \partial_\mu \varphi_\alpha \cong (\varphi_{\alpha,\bar z+\bar\mu} - \varphi_{\alpha,\bar z}) \Rightarrow 2\varphi^2_{\alpha,\bar z} - 2\varphi_{\alpha,\bar z+\bar\mu}\varphi_{\alpha,\bar z}$$

φ^4 - model and $O(2)$ - sigma model

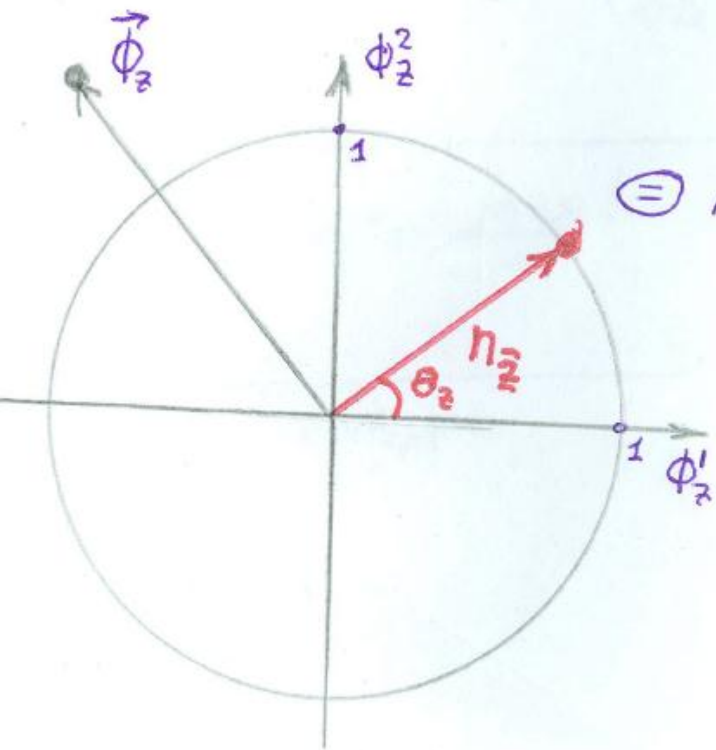
After reparametrization

$$\varphi_\alpha = \sqrt{\varkappa}\,\phi^\alpha \;;\; \Lambda = \Lambda(\lambda, \mu, \varkappa)$$

$$S = -\varkappa \sum_{\bar{z},\mu} \phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}+\bar{\mu}} + \sum_{\bar{z}} \left[\phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}} + \Lambda (\phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}} - 1)^2 \right]$$

$$Z = N \int \prod_{\bar{z}} d\phi^\alpha_{\bar{z}} \cdot e^{-S} = N \int \prod_{\bar{z}} d\phi^\alpha_{\bar{z}} \prod_{\bar{z}} e^{-[\phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}} + \Lambda(\phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}} - 1)^2]} *$$

$$* \, e^{\varkappa \sum_{\bar{z},\mu} \phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}+\bar{\mu}}} =$$



$$= N \int \prod_{z} d\mu(\phi^\alpha_{\bar{z}})\, e^{\varkappa \sum_{\bar{z},\mu} \phi^\alpha_{z} \phi^\alpha_{\bar{z}+\mu}}$$

$\downarrow$

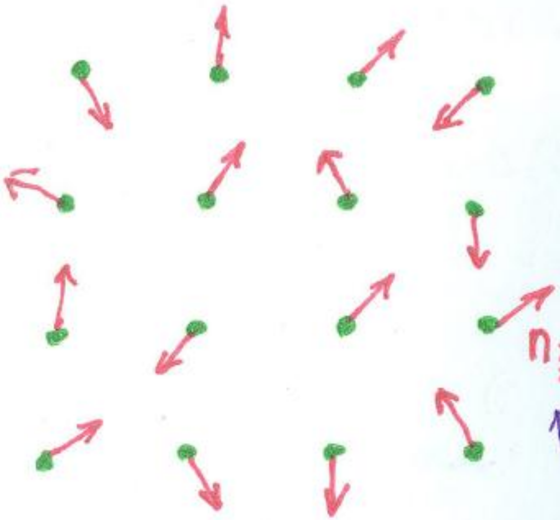
$$d\mu(\phi^\alpha_{\bar{z}}) = e^{-[\phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}} + \Lambda(\phi^\alpha_{\bar{z}} \phi^\alpha_{\bar{z}} - 1)^2]} \, d\phi^\alpha_{\bar{z}}$$

If $\Lambda \to \infty$: $\phi^\alpha_z \phi^\alpha_z \to 1$

and $d\phi^1_{\bar{z}} \, d\phi^2_z = R_{\phi_z} dR_{\phi_z} d\theta_{\phi_z} \Rightarrow d\theta_{\phi_z}$

$R_{\phi_z} \to 1$

O(2) sigma model (or XY model)



$n_{\bar{z}} : |n_z| = 1$

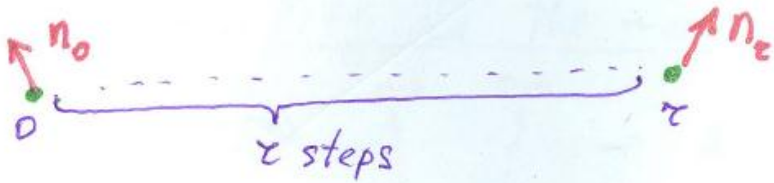
↑ like spin 1!

$$\underline{\underline{\mathcal{Z}}} = \int \prod_{\bar{z}} d\theta_{\bar{z}}\, e^{\varkappa \sum_{\bar{z},\mu} (n_{\bar{z}+\bar{\mu}} n_z)} =$$

$$= \int \prod_{\bar{z}} d\theta_{\bar{z}}\, e^{\varkappa \sum_{\bar{z},\mu} \cos(\theta_{\bar{z}+\bar{\mu}} - \theta_z)}$$

↑ This model like the Ising model but with O(2) symmetry!

Correlators!



$$\langle (n_0 n_\tau) \rangle = \langle \cos(\theta_0 - \theta_\tau) \rangle =$$

$$= \frac{1}{\mathcal{Z}} \int \prod_{z} d\theta_z \cos(\theta_0 - \theta_\tau)\, e^{\varkappa \sum_{\bar{z},\mu} \cos(\theta_{\bar{z}+\bar{\mu}} - \theta_{\bar{z}})}$$

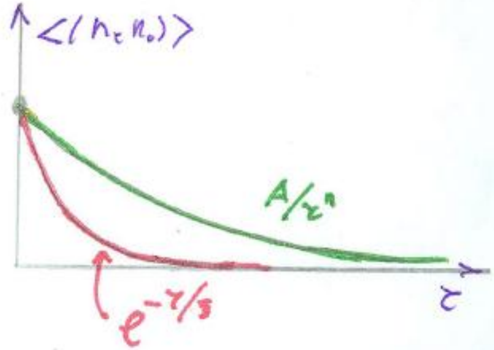
if $\tau = 1$: $\langle (n_0 n_\tau) \rangle = 1$

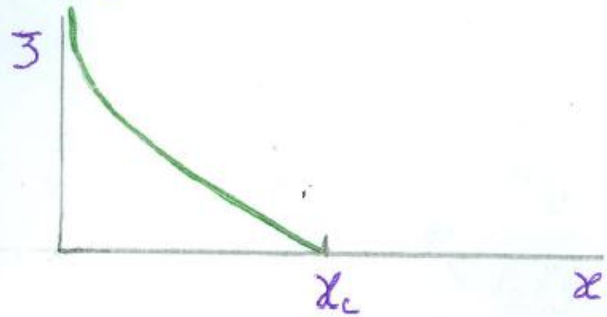
if $\tau \gg 1$: $\langle (n_0 n_\tau) \rangle \to 0$

BKT topological phase transition

If $\varkappa \to 0$: $\langle \cos(\theta_r - \theta_0) \rangle = \exp(-r/\xi)$, where ξ – correlation length

If $\varkappa \to \infty$: $\langle \cos(\theta_r - \theta_0) \rangle = \dfrac{A}{r^n}$

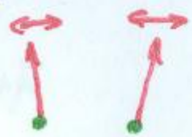




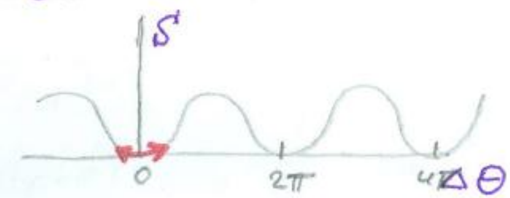
Phase transition $\longleftrightarrow$ instability $\longleftrightarrow$ non trivial topology

Instability in the model : qualitative view

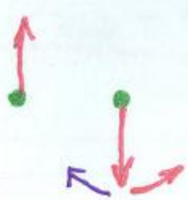
1. $\varkappa \to \infty$



$\to$ $S = -\cos \Delta\theta \sim$ min value

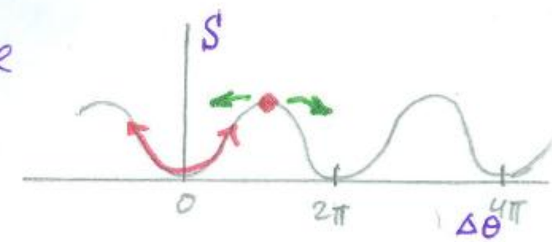


2. $\varkappa \to \varkappa_c$



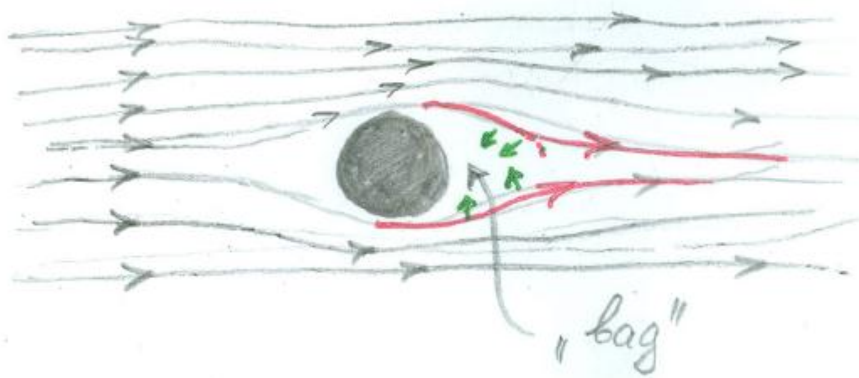
$\to$ $S = -\cos\Delta\theta \sim$ max value (can be)

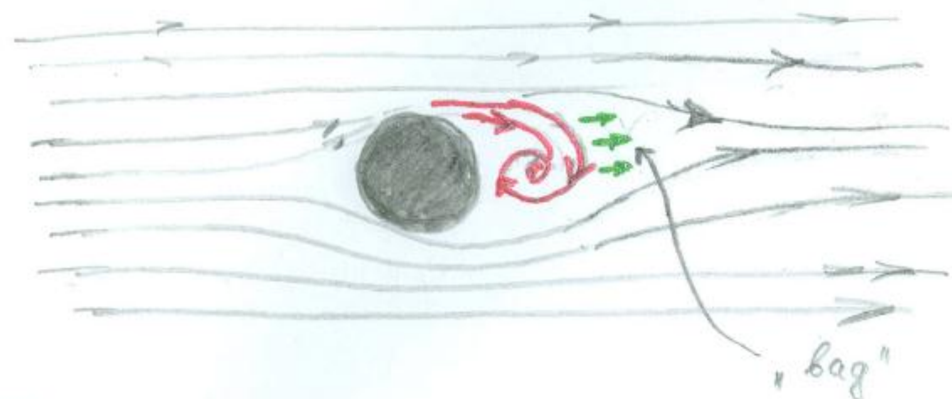
$\Delta\theta \sim \pi$ (can be)

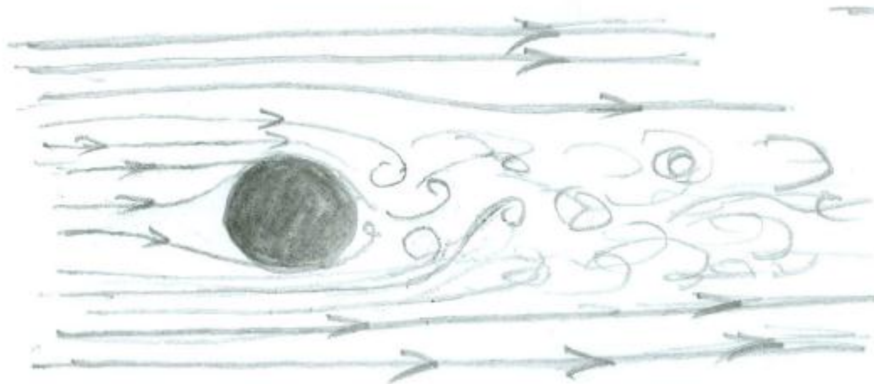


Hydrodynamical analogy : turbulence

Instability – Topological objects – phase transition







Villian aproximation, defectness, monopoles and vortecses

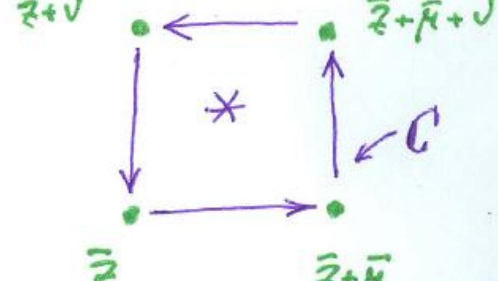
Rem.
$$\exp\left(\varkappa(\cos x-1)\right)\approx\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\varkappa}{2}(x+2\pi n)^2\right)$$

$$Z=\int\prod_{\bar z}d\theta_z\,\exp\left(\varkappa\sum_{\bar z\mu}\left(\cos(\theta_{\bar z+\bar\mu}-\theta_{\bar z})-1\right)\right)$$

$$\Downarrow$$

$$Z'=\sum_{n_{\bar z,\mu}}\int_{-\pi}^{\pi}\prod_{\bar z}d\theta_z\,\exp\left(-\frac{\varkappa}{2}\sum_{\bar z,\mu}\left(\theta_{\bar z+\bar\mu}-\theta_{\bar z}+2\pi n_{\bar z,\mu}\right)^2\right)$$

↖ like rigid rotator!



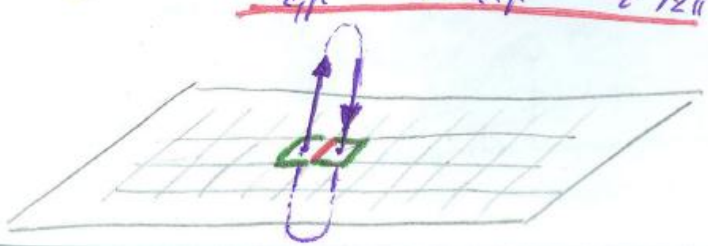
Let us consider

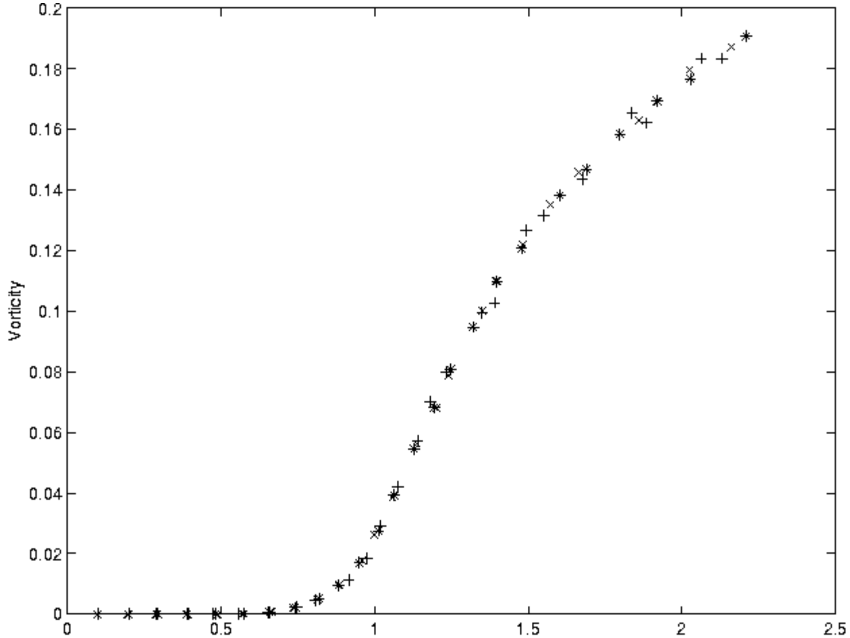
$$A_{\bar z,\mu}=\theta_{\bar z+\bar\mu}-\theta_{\bar z}$$

$$\oint_C A_{\bar z,\mu}\,dl_\mu=(\theta_{\bar z+\mu}-\theta_{\bar z})+(\theta_{\bar z+\bar\mu+\nu}-\theta_{\bar z+\mu})+$$
$$+(\theta_{z+\nu}-\theta_{z+\mu+\nu})+(\theta_z-\theta_{z+\nu})\equiv 0$$

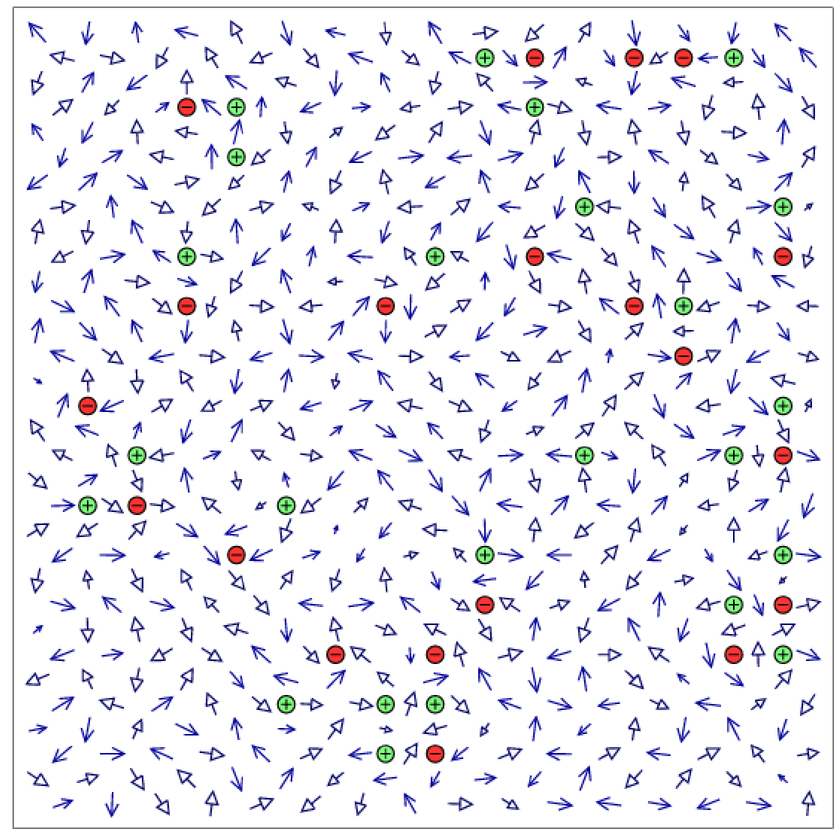
But $B_{\bar z,\mu}=(\theta_{\bar z+\bar\mu}-\theta_{\bar z})_{2\pi}\Rightarrow\oint_C B_{\bar z,\mu}\,dl_\mu=2\pi\left(n_{\bar z,\mu}+n_{x+\mu,\nu}+n_{x+\nu+\mu,-\mu}+n_{x+\nu,-\nu}\right)$

$$=2\pi\,Q_*$$





kT=0.22, E=-197.742, n1=44, nm=0, Z=0.605



Thermodynamics of vortices

$$Z = \int \prod_{\bar{z}} d\theta_{\bar{z}} \, e^{\varkappa \sum_{\bar{z},\mu} \cos(\theta_{\bar{z}+\mu} - \theta_{\bar{z}})}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\sum_{conf} \exp(-\beta E(conf)) \mapsto S = -\varkappa \sum_{\bar{z},\mu} \cos(\theta_{\bar{z}+\mu} - \theta_{\bar{z}})$$

$$\nearrow 1 - \frac{1}{2}(\theta_{z+\mu} - \theta_z)^2 + \dots$$

$$\Downarrow$$

$$S = Const + \int d^2 z \, \frac{\varkappa}{2} \partial_\mu \theta(z) \partial_\mu \theta(z)$$

Eq. of Motion : $\Delta \theta(\bar{z}) = 0$

in 2 dim

$$\theta(\bar{z}) \simeq \ln r \rightarrow \partial_\mu \theta(z) = \frac{Const}{r}$$

$$\oint \partial_\mu \theta \, e_\mu = 2\pi Q \rightarrow \partial\theta \simeq \frac{Q}{r}$$

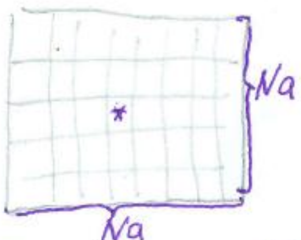
Stat. model interpritation:

$$\varkappa E_{vort} \simeq \frac{\varkappa}{2} \int d^2 z \, (\partial\theta)^2 = \frac{\varkappa}{2} \int_0^{2\pi} d\varphi \int_a^R r\,dr\,(\partial\theta)^2 =$$

$$= \varkappa\pi \int \left(\frac{Q}{r}\right)^2 r\,dr = \varkappa \pi Q \ln(R/a)$$

$$F_{vort} = E_{vort} - T S_{vort}$$

T: $1/\varkappa$; S_{vort}: entropy

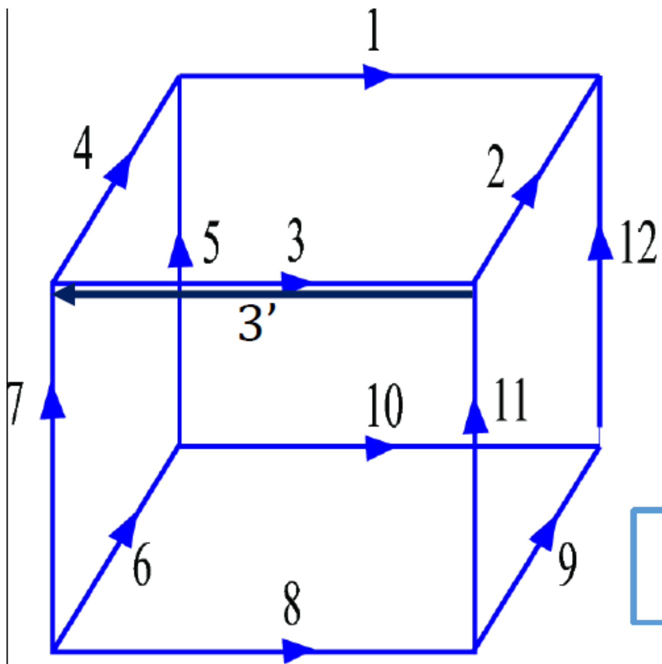


$$N_{position} = N^2 = (R/a)^2$$

$$S_{vort} = \ln N_{position} = 2 \ln(R/a)$$

$$F_{vort} = \pi Q \ln(R/a) - \frac{2}{\varkappa} \ln(R/a) \Rightarrow \varkappa_c = \frac{2}{\pi Q}$$

Compact lattice electrodynamics



$$u_{x,\mu} \equiv u_\mu(x),$$

$$u_{x,\mu} \in G$$

$$u_{x,-\mu} = u^*_{x-\mu,\mu}$$

$$\eta_x \equiv \eta(x),$$

$$u_{x,\mu} \to \eta_x u_{x,\mu} \eta_{x+\mu} \qquad (*)$$

$$u_p = u_{x,\mu} u_{x+\mu,\nu} u^*_{x+\nu,\mu} u^*_{x,\nu}$$

$$S = -\beta \sum_p u_p + S_{int}$$

$U(1)$

$$u_{x,\mu} = e^{i\theta_{x,\mu}},$$

$$\theta_{x,\mu} \in [-\pi, \pi)$$

$$u_{x,-\mu} = u^*_{x-\mu,\mu}$$

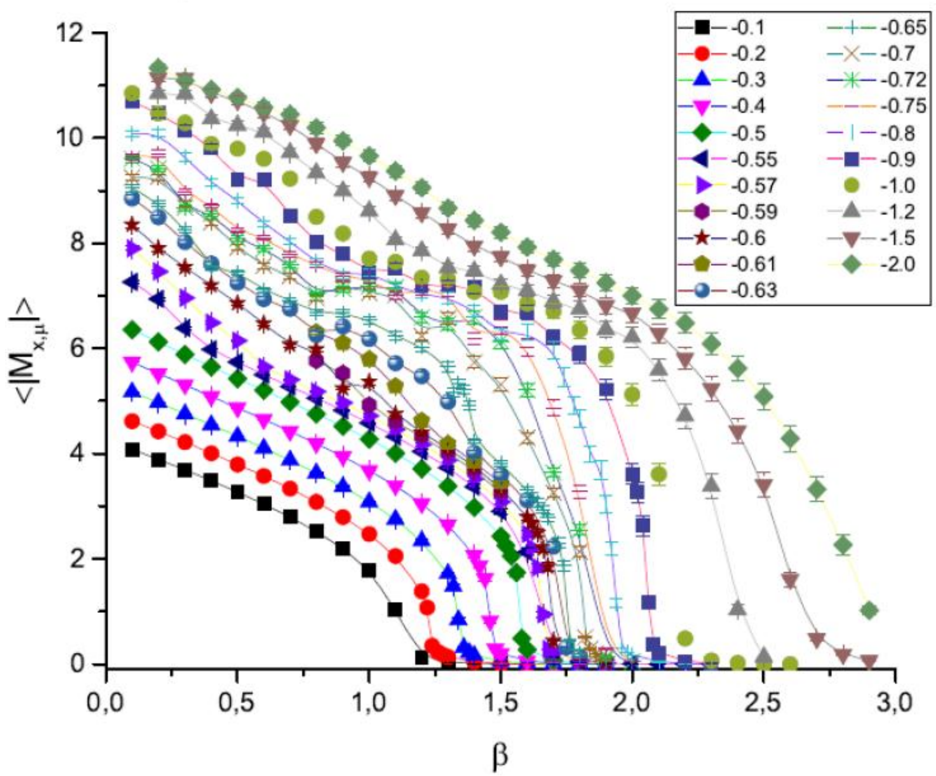
$$P(Conf) = \frac{1}{Z} e^{-S(Conf)}$$

Compact lattice electrodynamics

$$S = -\beta \sum_p u_p + \lambda \sum_{x,\mu} |M_{x,\mu}|$$

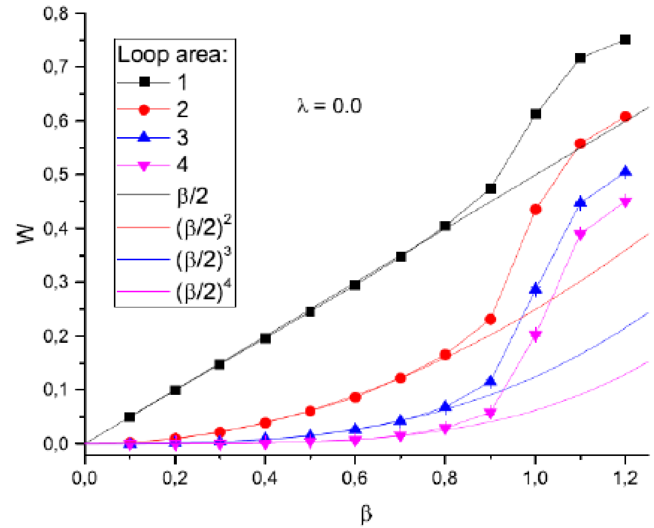
$$u_p = e^{i\theta_p}, \quad \theta_p \in [-a, a), a > \pi \;\Rightarrow\; \tilde{\theta}_p = \theta_p + 2\pi n_p$$

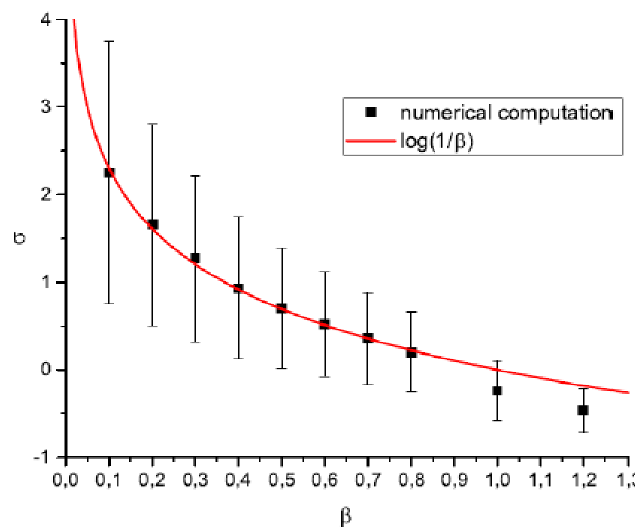
$$M_{x,\mu} = \sum_\nu \epsilon_{\mu\nu\rho\sigma}(n_{x+\nu,\rho\sigma} - n_{x,\rho\sigma})$$

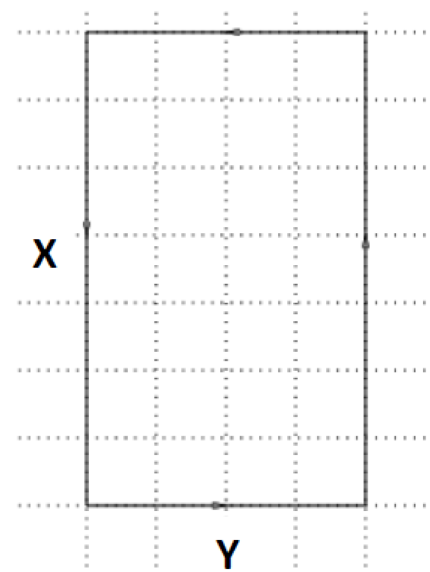


Compact lattice electrodynamics

$$\langle W \rangle = \beta^S = e^{S \ln \beta} = e^{-S \ln 1/\beta} = e^{-\sigma R \cdot T}$$

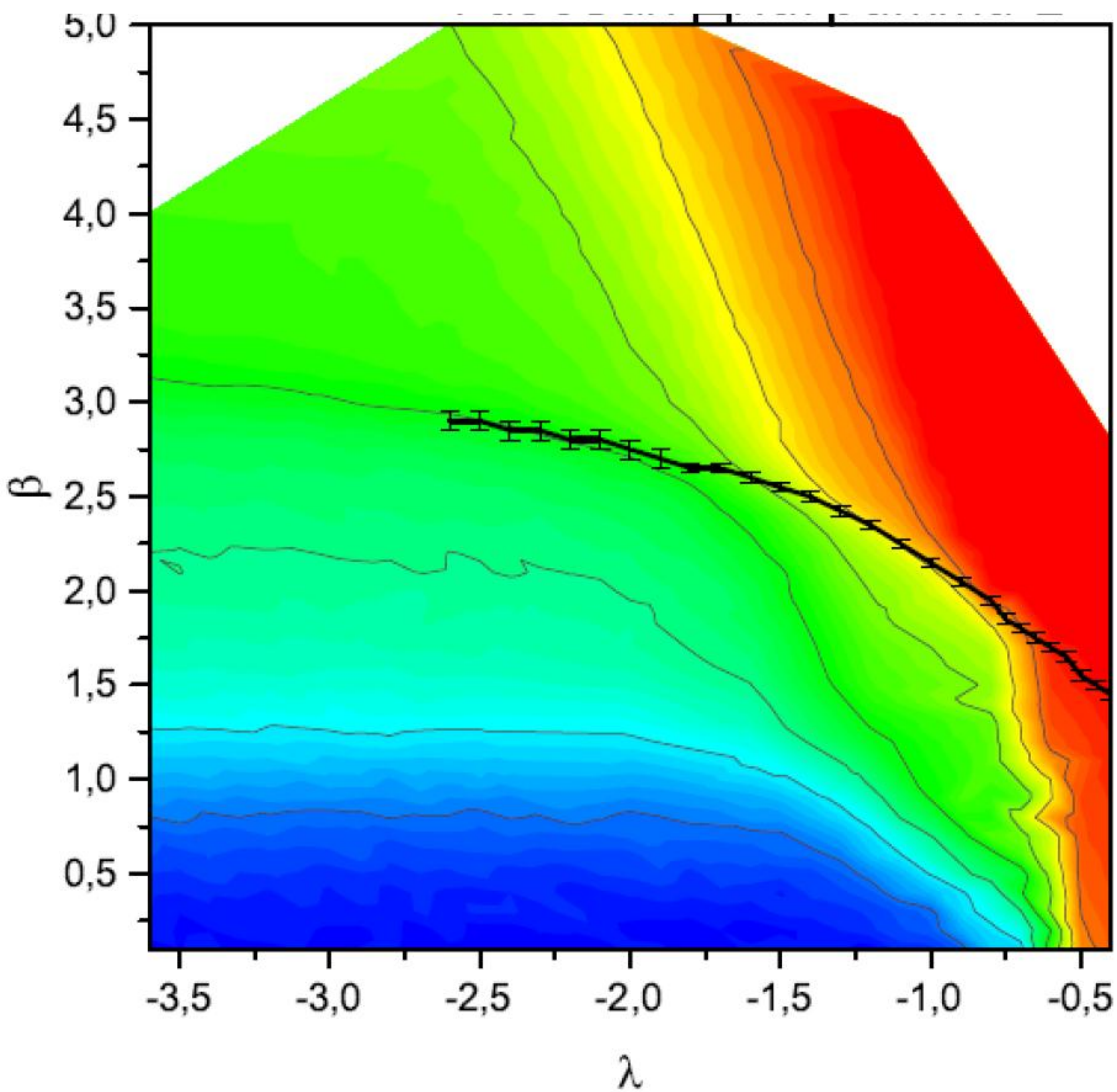






Compact lattice electrodynamics

$$C(d) = \left\langle \frac{1}{4} \sum_{\nu} M_{x,\mu} M_{x+d\nu,\mu} \right\rangle$$



Compact lattice electrodynamics